- June 13, 2005

Instructions: Compare your answers to the solutions posted here.

Part A: Multiple Choice

1. What are the critical numbers of the graph of $y = x^3 - 6x^2 + 9x + 3$?

(A) $x = 1; x = 3$ **(B)** $x = -1; x = 2$ **(C)** $x = 0; x = 2$

(D) $x = -3; x = -1$ **(E)** None of the above

Solution: **(A)**

$y = x^3 - 6x^2 + 9x + 3 \quad \Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 9 \quad \Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0 \quad \Rightarrow (x-1)(x-3) = 0 \quad \Rightarrow x = 1,\ x = 3$

2. What are the x –values at which the graph of $y = x^3 - 6x^2 + 9x + 3$ attains a relative maximum and a relative minimum?

(A) Relative maximum at $x = 3$, relative minimum at $x = 1$

(B) Relative maximum at $x = 1$, relative minimum at $x = 3$

(C) Relative maximum at $x = 0$, no relative minimum

(D) Relative minimum at $x = 0$, no relative maximum

Solution: **(B)**

Critical numbers are: $x = 1,\ x = 3$. Using the second derivative test:

$\frac{d^2y}{dx^2} = 6x - 12 \quad \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=1} = 6(1) - 12 = -6 < 0\ (\cap)$ So max. at $x = 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 6(3) - 12 = 6 > 0\ (\cup)$ So min. at $x = 3$

3. Identify the x –value(s) of the inflection point(s) of the graph of $y = x^3 - 6x^2 + 9x + 3$.

(A) $x = 0$ **(B)** $x = 1$ **(C)** $x = 2$ **(D)** $x = 3$ **(E)** None of the above

Solution: **(C)**

$\frac{d^2y}{dx^2} = 6x - 12 \quad \Rightarrow 6x - 12 = 0 \quad \Rightarrow x = 2$

Name: __Solution____

4. On what intervals is the graph of $y = x^3 - 6x^2 + 9x + 3$ concave up and concave down?

(A) Concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$

(B) Concave upward on $(-\infty, 2)$ and concave downward on $(2, \infty)$

(C) Concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$

(D) Concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$

Solution: (D)

Inflection point is at $x = 2$. So we check concavity before and after $x = 2$:

x	1	2	3
$\frac{d^2y}{dx^2}$	$+$	0	$-$
	$\cap$		$\cup$

5. Solve the equation $e^{x^2+x} = e^6$ for x..

(A) $x = 0$ or 1 **(B)** $x = 1$ or 2 **(C)** $x = -1$ or 3 **(D)** $x = 2$ or -3

Solution: (D)

Since the exponential function is one-to-one, then

$e^{x^2+x} = e^6 \quad \Rightarrow x^2 + x = 6 \quad \Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x-2)(x+3) = 0 \quad \Rightarrow x = 2$ or -3

6. Calculate the limit: $\lim_{x \to \infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7}$

(A) 0 **(B)** $\frac{1}{2}$ **(C)** 2 **(D)** ∞ **(E)** None of the above

Solution: (B)

Since the limit is at infinity and the numerator and denominator are polynomials of the same degree, then the limit is the quotient of the leading coefficients:

$$\lim_{x \to \infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7} = \frac{5}{10} = \frac{1}{2}$$

7. Which of the following is a pair of functions with the same derivative?

(A) $f(x) = 20x^4 + 300x + 250$ and $g(x) = 20x^4 + 300x^2 + 250$

(B) $f(x) = 20^5x^4 + 300x + 250^2$ and $g(x) = 20^5x^4 + 300x + 250^{10}$

(C) $f(x) = 20^3x^4 + x^e$ and $g(x) = 20^3x^4 + e^x$

(D) $f(x) = 209x^5$ and $g(x) = x^5$

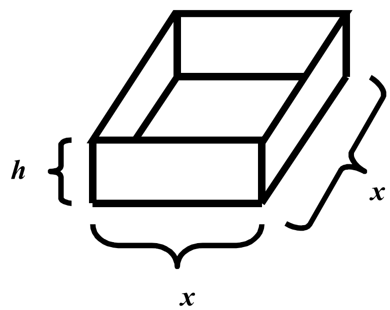
Solution: (B)

Since the derivative of a constant is zero, then the two functions that differ only by a constant have the same derivative. And so the correct choice is **(B)**

Name: __Solution____

The following word problem relates to numbers 8 to 10:

The figure below shows an open rectangular box with a square base. Consider the problem of finding the values of x (the length of each side of the base) and h (the height) for which the volume is 32 cubic feet ($32\,ft^3$) and the total surface area of the box is minimal. (The surface area S is the sum of the areas of the five faces of the box.)



8. Determine the **Objective/Primary Equation**.

(A) $x^2h = 32$ **(B)** $S = x^2h$ **(C)** $S = x^2 + 4xh$ **(D)** $x^2 + 4xh = 32$

Solution: **(C)**

We minimize surface area, S, so we make it the subject of the objective equation:

S = [Area of base] + [Area of sides] = $x^2 + 4xh$

9. Determine the **Constraint/Secondary Equation**.

(A) $x^2h = 32$ **(B)** $S = x^2h$ **(C)** $S = x^2 + 4xh$ **(D)** $x^2 + 4xh = 32$

Solution: **(A)**

Volume = length × width × height = $x \times x \times h = x^2h = 32$

10. Find the values of x and h for which the surface area of the box is a minimum.

(A) $x = 2\,ft, h = 4\,ft$ **(B)** $x = 4\,ft, h = 2\,ft$ **(C)** $x = 0\,ft, h = 0\,ft$

(D) $x = 8\,ft, h = \frac{1}{2}\,ft$ **(E)** None of the above.

Solution: **(B)**

We eliminate h and solve for x:

$x^2h = 32 \quad \Rightarrow h = 32x^{-2} \quad \Rightarrow S(x) = x^2 + 4x(32x^{-2}) = x^2 + 128x^{-1}$

$S'(x) = 2x - 128x^{-2} \quad \Rightarrow 2x - \frac{128}{x^2} = 0 \quad \Rightarrow 2x = \frac{128}{x^2}$

$x^3 = 64 \quad \Rightarrow x = 4 \quad h = \frac{32}{x^2} = \frac{32}{4^2} = \frac{32}{16} = 2$

Name: __Solution____

Part B: Free Response (Show all work!!!)

11. Find the **first and second derivatives** of the following functions. Simplify your answers.

a. $y = e^{5x}$

Solution:

$\frac{dy}{dx} = 5e^{5x}$

$\frac{d^2y}{dx^2} = 25e^{5x}$

b. $f(x) = 5x^2 - 2\ln x$

Solution:

$f'(x) = 10x - 2 \cdot \frac{1}{x} = 10x - 2x^{-1}$

$f''(x) = 10 + 2x^{-2} = 10 + \frac{2}{x^2}$

c. $y = e^{(x^2 - 5x + 4)}$

Solution:

$\frac{dy}{dx} = e^{(x^2 - 5x + 4)} \cdot \frac{d}{dx}[x^2 - 5x + 4] = e^{(x^2 - 5x + 4)}(2x - 5)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{(x^2 - 5x + 4)}\right] \cdot (2x - 5) + e^{(x^2 - 5x + 4)} \cdot \frac{d}{dx}\left[2x - 5\right]$ (Product Rule)

$\frac{d^2y}{dx^2} = \left(e^{(x^2 - 5x + 4)} \cdot \frac{d}{dx}[x^2 - 5x + 4]\right)(2x - 5) + e^{(x^2 - 5x + 4)}(2)$

$= e^{(x^2 - 5x + 4)}(2x - 5)^2 + 2e^{(x^2 - 5x + 4)}$

$\frac{d^2y}{dx^2} = e^{(x^2 - 5x + 4)}\left[(2x - 5)^2 + 2\right] = e^{(x^2 - 5x + 4)}(4x^2 - 20x + 27)$

12. Differentiate the following functions. **[Hint:** *Simplify the function first!***]**

a. $y = \ln\left(\frac{e^{x^2}}{x}\right)$

Solution:

$y = \ln\left(\frac{e^{x^2}}{x}\right) = \ln\left(e^{x^2}\right) - \ln x = x^2 - \ln x \quad \Rightarrow \frac{dy}{dx} = 2x - \frac{1}{x}$

b. $y = \ln\left(\frac{1}{e^{\sqrt{x}}}\right)$

Solution:

$y = \ln\left(\frac{1}{e^{\sqrt{x}}}\right) = \ln 1 - \ln\left(e^{\sqrt{x}}\right) = 0 - \sqrt{x} = -x^{1/2}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-1/2} = -\frac{1}{2\sqrt{x}}$

Name: __Solution____

13. In each case find the equation of the tangent line (in slope-intercept form) to the curve at the given point.

a. $y = x\ln x$ at the point $(1,0)$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}[x] \cdot \ln x + x \cdot \frac{d}{dx}[\ln x] = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Point is $(1,0)$ and slope is $\left.\frac{dy}{dx}\right|_{x=1} = \ln 1 + 1 = 1$

So equation is $y - 0 = 1(x-1) \quad \Rightarrow y = x - 1$

b. $y = xe^x$. at the point $(1,e)$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}[x] \cdot e^x + x \cdot \frac{d}{dx}[e^x] = e^x + xe^x$$

Point is $(1,e)$ and slope is $\left.\frac{dy}{dx}\right|_{x=1} = e^1 + 1e^1 = 2e$

So equation is $y - e = 2e(x-1) \quad \Rightarrow y = 2ex - e$

14. Use implicit differentiation to find $\frac{dy}{dx}$ for the graph of $x^2 - 3\ln y + y^2 = 10$

Solution:

$$\frac{d}{dx}\left[x^2\right] - \frac{d}{dx}\left[3\ln y\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}[10]$$

$$2x - 3 \cdot \frac{1}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \left(2y - \frac{3}{y}\right)\frac{dy}{dx} = -2x$$

$$\Rightarrow \left(\frac{2y^2 - 3}{y}\right)\frac{dy}{dx} = -2x \quad \Rightarrow \frac{dy}{dx} = -2x\left(\frac{y}{2y^2 - 3}\right)$$

$$\frac{dy}{dx} = \frac{-2xy}{2y^2 - 3} = \frac{2xy}{3 - 2y^2}$$

Some formulas and rules to remember:

1. $a^u = a^v$ if and only if $u = v$ **2.** $\ln e^x = x$ and $e^{\ln x} = x$

3. $\ln(MN) = \ln M + \ln N$, if $M > 0$ & $N > 0$ **4.** $\ln(\frac{M}{N}) = \ln M - \ln N$, if $M > 0$ & $N > 0$

5. $\ln\left(M^k\right) = k\ln M$ if $M > 0$ **6.** $\frac{d}{dx}[e^{g(x)}] = e^{g(x)} \cdot g'(x)$

7. $\frac{d}{dx}[\ln(g(x))] = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$ **8.** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

9. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$ **10.** $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x)$