1. (a) (5 pts) Use Newton's method, with $x_1 = 1$, to find x_2, the second approximation to the root of the equation $x^4 = 5x - 2$

(b) (7 pts) What is the smallest possible perimeter for a rectangle with area equal to 100m^2?

Solution:

(a) We approximate a root to the equation $f(x) = 0$ where $f(x) = x^4 - 5x + 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^4 - 5x_1 + 2}{4x_1^3 - 5} = 1 - \frac{1-5+2}{4-5} = 1 - 2 = -1$$

(b) We wish to minimize $P = 2x + 2y$ subject to the constraint $xy = 100$. So, $y = 100/x$ and so we wish to minimize $P(x) = 2x + 200/x$, $x > 0$. Now note that $P'(x) = 2 - \frac{200}{x^2}$ and $P'(x) = 0$ implies $x = \pm 10$, and we need $x > 0$, so $x = 10$. Note that $P''(x) = 400/x^3$ and $P''(10) > 0$ so we have a minimum. Now $x = 10$ implies $y = 10$ and so the smallest possible perimeter is $\boxed{P = 40\text{m}}$

2. (a) (5 pts) Use a Riemann sum with a regular partition, 6 subintervals, and left endpoints to approximate the net-area of the region bounded by $f(x) = x^2 - 5x + 4$, $0 \le x \le 6$, and the x-axis.

(b) (10 pts) Given that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \text{and} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Use the **limit definition of the integral** with right endpoints and a regular partition, to evaluate the integral $\int_0^6 (x^2 - 5x + 4)\, dx$. Simplify your answer!

Solution:

(a) Note that $\Delta x_i = \Delta x = (6-0)/6 = 1$ and $x_{i-1} = 0 + (i-1)\Delta x = i - 1$ and here $f(x) = x^2 - 5x + 4$, thus

$$\begin{aligned}\int_0^6 (x^2 - 5x + 4)\, dx \approx \sum_{i=1}^{6} f(x_i^*)\Delta x_i \overbrace{=}^{x_i^* = x_{i-1}} & \sum_{i=1}^{6} f(i-1) \cdot 1 \\ = & \; f(0) + f(1) + f(2) + f(3) + f(4) + f(5) \\ = & \; 4 + 0 - 2 - 2 + 0 + 4 = 4\end{aligned}$$

(b) Here, $\Delta x = (6-0)/n = 6/n$ and $x_i = 0 + i\Delta x = 6i/n$ and so,

$$\begin{aligned}\int_0^6 (x^2 - 5x + 4)\, dx \overbrace{=}^{x_i^* = x_i} & \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{36i^2}{n^2} - \frac{30i}{n} + 4\right)\frac{6}{n} \\ = & \lim_{n\to\infty} \left[\frac{216}{n^3}\sum_{i=1}^{n} i^2 - \frac{180}{n^2}\sum_{i=1}^{n} i + \frac{24}{n}\sum_{i=1}^{n} 1\right] \\ = & \lim_{n\to\infty} \left[\frac{216}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{180}{n^2}\frac{n(n+1)}{2} + \frac{24}{n}n\right] \\ = & \lim_{n\to\infty} \left[\frac{432n^3 + (\text{lower order terms})}{6n^3} - \frac{180n^2 + (\text{lower order terms})}{2n^2} + 24\right] \\ = & \frac{432}{6} - \frac{180}{2} + 24 = 72 - 90 + 24 = 6\end{aligned}$$

3. (18 pts) Evaluate the following integrals, justify your answer:

(a) $\int_{0.5}^{0} \sqrt{1-4x^2}\,dx$ (b) $\int (x^\pi + \pi x + \pi^2)\,dx$ (c) $\int_{-1}^{1} (x^2+1)^{1/3} x^7\,dx$

Solution:

(a) Note that $\int_{0.5}^{0} \sqrt{1-4x^2}\,dx = -2\int_{0}^{1/2} \sqrt{\frac{1}{4}-x^2}\,dx$ and if we interpret $\int_{0}^{1/2} \sqrt{\frac{1}{4}-x^2}\,dx$ geometrically, then this is one quarter the area of a circle of radius 1/2, so

$$\int_{0.5}^{0} \sqrt{1-4x^2}\,dx = -2 \underbrace{\left(\int_{0}^{1/2} \sqrt{\frac{1}{4}-x^2}\,dx\right)}_{\text{1/4 the area of a circle with r=1/2}} = -2\cdot\frac{\pi(1/2)^2}{4} = -\frac{\pi}{8}$$

(b) $\int (x^\pi + \pi x + \pi^2)\,dx = \frac{x^{\pi+1}}{\pi+1} + \frac{\pi x^2}{2} + \pi^2 x + C$

(c) Note that $f(x) = x^7(x^2+1)^{1/3}$ is an odd function, so $\int_{-1}^{1} (x^2+1)^{1/3} x^7\,dx = 0.$

4. (a) (8 pts) The position function of Annie the ant is given by $s(t) = \int_{t^2}^{3t+1} \sin(x^4)\,dx$, where t is in seconds, find Annie's velocity at any time t.

(b) (7 pts) Find a function $f(t)$ such that $\int_{1}^{x} \frac{f(t)}{t^2}\,dt = 2\sqrt{x} - 2.$

Solution:

(a) Note that $v(t) = s'(t)$ and $s(t) = \int_{t^2}^{3t+1} \sin(x^4)\,dx = \int_{t^2}^{0} \sin(x^4)\,dx + \int_{0}^{3t+1} \sin(x^4)\,dx$ and so

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[-\int_{0}^{t^2} \sin(x^4)\,dx + \int_{0}^{3t+1} \sin(x^4)\,dx\right] \\ &= -2t\sin(t^8) + 3\sin((3t+1)^4) \end{aligned}$$

so Annie's velocity is $\boxed{v(t) = -2t\sin(t^8) + 3\sin((3t+1)^4)}$

(b) Taking the derivative of both sides yields $\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$ and solving for $f(x)$ yields

$$f(x) = x^2/\sqrt{x} = x^{3/2} \text{ and so } \boxed{f(t) = t^{3/2}}$$

5. (a) (8 pts) If g is a continuous function on the interval $[0,1]$, show that $\int_{0}^{1} g(1-x)\,dx = \int_{0}^{1} g(x)\,dx$

(b) (8 pts) Suppose $\int_{106}^{53} 2f(m)\,dm = -4\pi$ and $\int_{53}^{90} \frac{f(k)}{2}\,dk = \frac{\pi}{2}$, find $\int_{90}^{106} f(b)\,db.$

Solution:

(a) Let $u = 1-x$ then $du = -dx$ and we get $\int_{0}^{1} g(1-x)\,dx = -\int_{1}^{0} g(u)\,du = \int_{0}^{1} g(u)\,du.$

(b) Note that $\int_{53}^{106} f(m)\,dm = 2\pi$, and $\int_{53}^{90} f(k)\,dk = \pi$ and, $\int_{53}^{106} f(b)db = \int_{53}^{90} f(b)db + \int_{90}^{106} f(b)db$ thus,

$$\int_{90}^{106} f(b)db = \int_{53}^{106} f(b)db - \int_{53}^{90} f(b)db = \int_{53}^{106} f(m)dm - \int_{53}^{90} f(k)dk = 2\pi - \pi = \pi.$$

6. (24 pts) Evaluate the following integrals, justify your answer:

(a) $\displaystyle\int_0^4 \frac{x\ dx}{\sqrt{1+2x}}$ (b) $\displaystyle\int_0^3 |x^2-4|\ dx$ (c) $\displaystyle\int x^2\left(1+x^{3/2}\right)^9\ dx$

Solution:

(a) If we let $u = 1+2x$ then $du = 2dx$ and $x = (u-1)/2$ and note if $x = 0$ we have $u = 1$ and when $x = 4$, we have $u = 9$, so we have

$$\int_0^4 \frac{x\ dx}{\sqrt{1+2x}} = \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\ du = \frac{1}{4}\int_1^9 (u^{1/2}-u^{-1/2})\ du = \frac{1}{4}\left[\frac{2}{3}u^{3/2}-2u^{1/2}\right]\Bigg|_1^9 = \frac{1}{4}\left[12+\frac{4}{3}\right] = \frac{10}{3}$$

(b) Note that $x^2-4<0$ for $-2<x<2$, and so

$$\begin{aligned}\int_0^3 |x^2-4|\ dx &= \int_0^2 -(x^2-4)\ dx + \int_2^3 (x^2-4)\ dx \\ &= \left(-\frac{x^3}{3}+4x\right)\Bigg|_0^2 + \left(\frac{x^3}{3}-4x\right)\Bigg|_2^3 \\ &= \frac{16}{3}+\frac{7}{3} = \frac{23}{3}\end{aligned}$$

(c) If we let $u = 1+x^{3/2}$ then $du = \frac{3}{2}x^{1/2}dx$ and $x^{3/2} = u-1$ and so,

$$\int x^2\left(1+x^{3/2}\right)^9\ dx \underbrace{=}_{x^2=x^{3/2}\cdot x^{1/2}} \int x^{3/2}\left(1+x^{3/2}\right)^9\ x^{1/2}dx = \frac{2}{3}\int (u-1)u^9\ du = \frac{2}{3}\int (u^{10}-u^9)\ du$$

thus we have, $\displaystyle\frac{2}{3}\int (u^{10}-u^9)\ du = \frac{2}{3}\left(\frac{u^{11}}{11}-\frac{u^{10}}{10}+C\right) = \frac{2}{33}(1+x^{3/2})^{11}-\frac{1}{15}(1+x^{3/2})^{10}+C.$