1. (21 pts) Each part is worth 7 points.

   (a) Find the derivative of $g(x) = \cot(x^2) + \cot^2(x)$

   (b) Find an equation of the tangent line to the curve $y = \dfrac{|x|}{\sqrt{2-x^2}}$ when $x = 1$.

   (c) Find an equation of the tangent line to the curve $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

   Solution:

   (a) $f'(x) = -\csc^2(x^2) \cdot 2x + 2\cot(x) \cdot -\csc^2(x) = -2x\csc^2(x^2) - 2\cot(x)\csc^2(x)$

   (b) Note that if $x > 0$ then $y = \dfrac{x}{\sqrt{2-x^2}}$ and $y' = \dfrac{2}{(2-x^2)^{3/2}}$, so $y'(1) = 2$ and $y(1) = 1$, so the equation of the tangent line is $y = 1 + 2(x-1) = -1 + 2x$

   (c) Using implicit differentiation we have $2x + y + xy' + 2yy' = 0$, so $y' = \dfrac{-(2x+y)}{x+2y}$ and $y'(1,1) = -1$ and so the equation of the tangent line is $y = 1 + (-1)(x-1) = 2 - x$.

2. (a) (10 pts) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ at $x = 3$.

   (b) (10 pts) The radius of a circular disk is given as 4 cm with an error in measurement of ± 0.2 cm. Use differentials to determine the percentage error in the calculated area of the disk.

   Solution:

   (a) Note that $f'(x) = \dfrac{-x}{\sqrt{25-x^2}}$ and $L(x) = f(3) + f'(3)(x-3) = 4 - \frac{3}{4}(x-3) = \frac{25}{4} - \frac{3}{4}x$

   (b) Note that $A = \pi r^2$ and so $dA = 2\pi r dr$ and thus

$$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r} = \frac{2(\pm 0.2)}{4} = \pm 0.1$$

   so the relative error is ± 0.1 and the percentage error is $\pm 10\%$.

3. (10 pts) Find the absolute maximum and absolute minimum values of $f(x) = x\sqrt{4-x^2}$ on $[-1, 2]$

   Solution:

   Here $f'(x) = \dfrac{4-2x^2}{\sqrt{4-x^2}}$ so $f'(x) = 0$ when $x = \pm\sqrt{2}$ and $f'(x)$ is undefined when $x = \pm 2$ so the critical points are $x = \pm\sqrt{2}, \pm 2$.

   The critical points in our interval are $x = \sqrt{2}, 2$, and $f(-1) = -\sqrt{3}$, $f(2) = 0$ and $f(\sqrt{2}) = 2$.

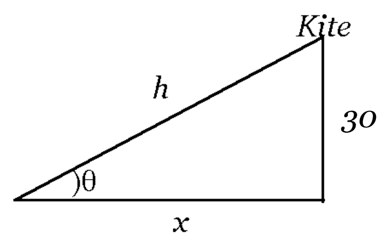
   So the absolute minimum value is $-\sqrt{3}$ and the absolute maximum value is 2.

4. (10 pts) A kite 30 ft above the ground moves horizontally at a speed of 10 ft/s. At what rate is the angle between the kite-string and the ground decreasing when 50 ft of string has been let out?

Solution:

Let x be the horizontal distance the kite has traveled and let h be the length of string that has been let out. Then we know $dx/dt = 10$ft/s. Now note that if θ is the angle that the string makes with the ground then we have $\cot(\theta) = x/30$ and so

$$\frac{d\theta}{dt} = -\sin^2(\theta) \cdot \frac{dx}{dt} \cdot \frac{1}{30} = -\left(\frac{30}{50}\right)^2 \cdot 10 \cdot \frac{1}{30} = -\frac{3}{25}\text{rad/s}$$



5. (21 pts) Each part is worth 7 points.

(a) Find constants F and M such that the function $y = F\sin(x) + M\cos(x)$ satisfies the differential equation $y'' + y' = \sin(x)$.

(b) Suppose $f(x)$ is differentiable for all values of x. Let $B(x) = f(x) \circ \cos(x)$, find an expression for $B''(x)$.

(c) Use the Mean Value Theorem on the function $f(x) = \sqrt{1+x}$, for $0 \le x \le 1$, to show that $\sqrt{2} < 3/2$.

Solution:

(a) Note that $y' = F\cos(x) - M\sin(x)$ and $y'' = -F\sin(x) - M\cos(x)$ and so,

$$y'' + y' = (-F - M)\sin(x) + (F - M)\cos(x) = \sin(x)$$

so $-F - M = 1$ and $F - M = 0$, and so $F = M = -\frac{1}{2}$.

(b) Here $B(x) = f(\cos(x))$ and so, $B'(x) = f'(\cos(x)) \cdot -\sin(x)$, and

$$B''(x) = f''(\cos(x))\sin^2(x) - f'(\cos(x))\cos(x)$$

(c) Note that $f'(x) = \frac{1}{2\sqrt{1+x}}$ and so by the Mean Value Theorem, there exists a c in $(0,1)$ such that

$$\frac{f(1) - f(0)}{1 - 0} = f'(c), \text{ that is, } \sqrt{2} - 1 = \frac{1}{2\sqrt{1+c}} \text{ for some } c \text{ in } (0,1)$$

and so $\sqrt{2} = 1 + \frac{1}{2\sqrt{1+c}}$ and note that $\frac{1}{2\sqrt{1+c}} < \frac{1}{2}$ since $\sqrt{1+c} > 1$ and so,

$$\sqrt{2} = 1 + \frac{1}{2\sqrt{1+c}} < 1 + \frac{1}{2} = \frac{3}{2}$$

and so we have $\sqrt{2} < 3/2$.

6. (18 pts) Given $f(x) = \dfrac{1+2x^2}{x^2-1}$, $f'(x) = \dfrac{-6x}{(x^2-1)^2}$, and $f''(x) = \dfrac{6(3x^2+1)}{(x^2-1)^3}$

(a) Find the intervals on which $f(x)$ is increasing or decreasing and find any local maximum and minimum points of $f(x)$.(Give the x and y coordinate of any local max or min's.) Justify your answers.

(b) Describe the concavity of $f(x)$, are there any inflection points? If so, what is the coordinate of the inflection point? (Give the x and y coordinate of any inflection points.) Justify your answers.

(c) In your blue book, clearly sketch the graph of $f(x)$. Be sure to LABEL all horizontal and vertical asymptotes, intercepts, local maximums and minimums.

Solution:

(a) Increasing on $(-\infty,-1)\cup(-1,0)$; decreasing on $(0,1)\cup(1,\infty)$; local maximum at $(0,-1)$

(b) Concave up on $(-\infty,-1)\cup(1,\infty)$; concave down on $(-1,1)$; there are no inflection points

(c) The graph: