1. (30 pts) Use the **limit definition of the derivative** to find:

   (a) $f'(x)$ given $f(x) = \sqrt{x^2+9}$

   (b) $f'(0)$ given $f(x) = 5x^4 + 3x + 20$

   (c) $\frac{d^2f}{dx^2}$ given $\frac{d}{dx}f(x) = \frac{1}{10x+6}$

Solution:

(a) We have

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{(x+h)^2+9}-\sqrt{x^2+9}}{h}\cdot\frac{\sqrt{(x+h)^2+9}+\sqrt{x^2+9}}{\sqrt{(x+h)^2+9}+\sqrt{x^2+9}} \\ &= \lim_{h\to 0}\frac{((x+h)^2+9)-(x^2+9)}{h(\sqrt{(x+h)^2+9}+\sqrt{x^2+9})} \\ &= \lim_{h\to 0}\frac{2xh+h^2}{h(\sqrt{(x+h)^2+9}+\sqrt{x^2+9})} = \frac{2x}{2\sqrt{x^2+9}} = \frac{x}{\sqrt{x^2+9}} \end{aligned}$$

(b) We have

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{5x^4+3x+20-20}{x} = \lim_{x\to 0}\frac{5x^4+3x}{x} = \lim_{x\to 0}5x^3+3 = 3$$

(c) We have

$$\begin{aligned} f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{10(x+h)+6}-\frac{1}{10x+6}}{h} \\ &= \lim_{h\to 0}\frac{10x+6-(10(x+h)+6)}{(10(x+h)+6)(10x+6)}\cdot\frac{1}{h} = \lim_{h\to 0}\frac{-10h}{(10(x+h)+6)(10x+6)}\cdot\frac{1}{h} \\ &= \frac{-10}{(10x+6)^2} \end{aligned}$$

2. Suppose the position function of a particle in motion is given by $s(t) = 5t^4 + 3t + 20$ ft, where t is in seconds.

   (a) (8 pts) Find the average rate of change of the position of the particle from $t = 0$ seconds to $t = 2$ seconds.

   (b) (6 pts) Find the instantaneous rate of change of the position of the particle when $t = 0$ seconds.

Solution:

(a) Here we have

$$\text{Average Rate of Change} = \frac{s(2)-s(0)}{2-0} = \frac{106-20}{2} = \frac{86}{2} = 43 \text{ ft/s}$$

(b) From #1(b) we have $s'(0) = 3$ ft/s

3. (18 pts) Let $f(x) = x^2 + 1$, $m(x) = x$ and $b(x) = \sqrt{x^2 - 1}$,

   (a) Find $(f \circ m \circ b)(x)$ and state the domain.
   (b) Find $(f + m)(x)$ and state the domain.
   (c) Find $\left(\dfrac{b}{m}\right)(x)$ and state the domain.

Solution:

(a) $(f \circ m \circ b)(x) = f(m(b(x))) = x^2$ with domain $(-\infty, -1] \cup [1, \infty)$

(b) $(f + m)(x) = x^2 + 1 + x$ with domain $(-\infty, \infty)$

(c) $\left(\dfrac{b}{m}\right)(x) = \dfrac{\sqrt{x^2 - 1}}{x}$ with domain $(-\infty, -1] \cup [1, \infty)$

4. (20 pts) Justify your answers for all the questions below. Consider the function $f(x) = \dfrac{1}{x+1} + \dfrac{2x}{|x|}$, does $f(x)$ have any:

   (a) *vertical* asymptotes? If so, what are they?
   (b) *horizontal* asymptotes? If so, what are they?
   (c) *removable* discontinuities? If so, what are they?
   (d) *jump* discontinuities? If so, what are they?

Solution:

(a) Note, $\lim\limits_{x \to -1^-} f(x) = -\infty$ and $\lim\limits_{x \to -1^+} f(x) = \infty$, and so there is a vertical asymptote at $x = -1$.

(b) Note, $\lim\limits_{x \to \infty} f(x) = 2$ and $\lim\limits_{x \to -\infty} f(x) = -2$, so the horizontal asymptotes are $y = 2$ and $y = -2$.

(c) No, there are NO removable discontinuities.

(d) Yes, there is a jump discontinuity at $x = 0$ since $\lim\limits_{x \to 0^-} f(x) = -1$ and $\lim\limits_{x \to 0^+} f(x) = 3$.

5. (18 pts) Let a, b and c be constants and consider the function $f(x)$ where

$$f(x) = \begin{cases} x + 6, & x \leq 0 \\ cx^2 + bx + a, & 0 < x < 1 \\ 7x + c, & x \geq 1 \end{cases}$$

   (a) Find all values of a, b and c for which $f(x)$ will be *continuous* at $x = 0$ and $x = 1$?
   (b) For what values of a, b and c will $f(x)$ be *differentiable* at $x = 0$ and $x = 1$?
   (c) Find the equation of the tangent line to $f(x)$ at $x = -1$.

Solution:

(a) For just continuity, we need $a = 6$ and $b = 1$ and c can have any value.

(b) Note that,

$$f'(x) = \begin{cases} 1, & x < 0 \\ 2cx + 1, & 0 < x < 1 \\ 7, & x > 1 \end{cases}$$

and so for differentiability we need $a = 6$, $b = 1$ and $c = 3$.

(c) Here $f'(-1) = 1$ and $(-1, f(-1)) = (-1, 5)$ and so the equation is $y = 5 + (x + 1) = x + 6$.