1. [14 points] Indicate if each of the following is true or false by circling the correct answer. Justify your answer.

a. [2 points] If $\int_0^\infty f(x)dx$ is divergent then $\int_1^\infty f(x)dx$ is also divergent.

True **[False]**

Solution: $\int_1^\infty f(x)dx$ can be convergent while $\int_0^1 f(x)dx$ is divergent. Example $f(x) = \frac{1}{x^2}$.

b. [2 points] If the median of a density function $p(t)$ is 0, then $p(t)$ is an even function.

True **[False]**

Solution: Example: $p(x) = \begin{cases} .5 & -1 \le x \le 0 \\ .25 & 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$ has median 0 but it is not symmetric around the y axis.

c. [4 points] A curve is parametrized by the functions $x(t) = 1 - t^2$ and $y(t) = t^4 + 3t^2$ for $0 \le t \le 1$. The concavity of the graph of the parametric curve is positive for $0 < t < 1$.

[True] False

Solution:

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{y'}{x'}\right)'}{x'} = \frac{\left(\frac{4t^3+6t}{-2t}\right)'}{-2t} = \frac{(-2t^2-3)'}{-2t} = \frac{-4t}{-2t} = 2.$$

d. [2 points] In polar coordinates, the coordinates $(2, \frac{\pi}{3})$ and $(-2, \frac{7\pi}{3})$ represent the same point.

True **[False]**

Solution: $(2, \frac{\pi}{3})$ is in quadrant I and $(-2, \frac{7\pi}{3})$ is in quadrant III

e. [2 points] If $P(t)$ is a cumulative distribution function then $\int_{-\infty}^\infty P(t)dt$ converges.

True **[False]**

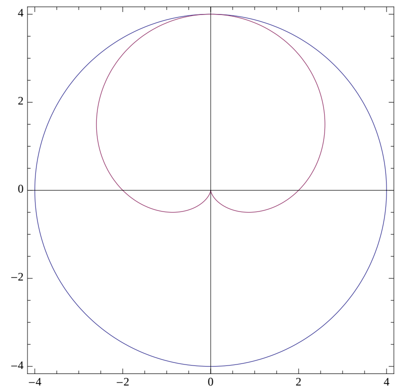
Solution: $P(t)$ is an increasing functions and $\lim_{t\to\infty} P(t) = 1$ hence $\int_{-\infty}^\infty P(t)dt$ diverges.

f. [2 points] The solutions to the differential equation $\frac{dy}{dx} = 1 + y^2 + 3x^2$ are increasing at every point.

[True] False

Solution: $y' = 1 + y^2 + 3x^2 > 0$ hence y is an increasing function.

2. [14 points] The graph of the circle $r = 4$ and and the cardioid $r = 2\sin\theta - 2$ are shown below.



a. [3 points] Write a formula for the area inside the circle and outside the cardioid in the first quadrant.

Solution: Area of the quarter of a circle= 4π
Area of cardioid=$\int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$
Area=$4\pi - \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

b. [7 points] At what angles $0 \leq \theta < 2\pi$ is the minimum value of the y coordinate on the cardioid attained? No credit will be given for answers without proper mathematical justification.

Solution:
$y(\theta) = (2\sin\theta - 2)\sin\theta$

$y'(\theta) = 2\cos\theta\sin\theta + (2\sin\theta - 2)\cos\theta = 4\cos\theta\sin\theta - 2\cos\theta$

Critical points $(4\sin\theta - 2)\cos\theta = 0$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$ then $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.
Minimum y coordinate at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

c. [4 points] Write an integral that computes the value of the length of the piece of the cardioid lying below the x-axis.

Solution:
$x(\theta) = (2\sin\theta - 2)\cos\theta$ $x'(\theta) = 2\cos^2\theta - (2\sin\theta - 2)\sin\theta$

$L = \int_0^{\pi} \sqrt{(2\cos^2\theta - (2\sin\theta - 2)\sin\theta)^2 + (4\cos\theta\sin\theta - 2\cos\theta)^2} d\theta$

3. [13 points]
The phones offered by a cell phone company have some chance of failure after they are activated. Suppose that the density function $p(t)$ describing the time t in years that one of their phones will fail is

$$p(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

a. [5 points] Find the cumulative distribution function $P(t)$ of $p(t)$.

Solution: $P(t) = \int_0^t \lambda e^{-\lambda t} dt = -e^{-\lambda t}|_0^t = 1 - e^{-\lambda t}$

$$P(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

b. [4 points] If the probability of a cell phone failing within a year and a half is $\frac{2}{5}$, find the value of λ.

Solution: $\int_0^{1.5} \lambda e^{-\lambda t} dt = 1 - e^{-1.5\lambda}$
$1 - e^{-1.5\lambda} = \frac{2}{5}$ then $\lambda = -\frac{\ln(\frac{3}{5})}{1.5} = .34$

c. [4 points] The cell phone company offers its clients a replacement phone after two years if they sign a new contract. What is the probability that the client will not have to replace his or her phone before the company will give him or her a new one?

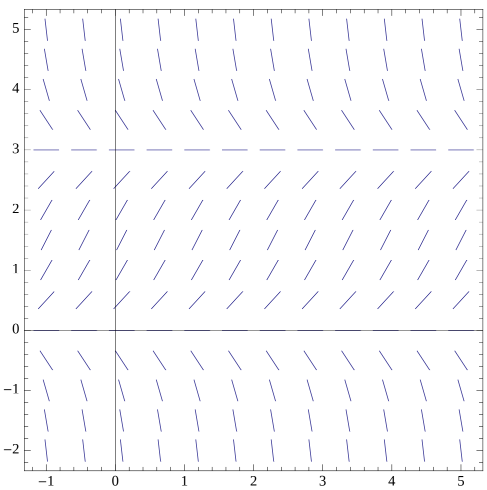
Solution:
$\int_2^{\infty} \lambda e^{-\lambda t} dt = \lim_{b\to\infty} -e^{-\lambda t}|_2^b = \lim_{b\to\infty} e^{-.68} - e^{-.34b} = e^{-.68} = .506$
or
$1 - P(2) = e^{-.68} = 506.$

4. [15 points]

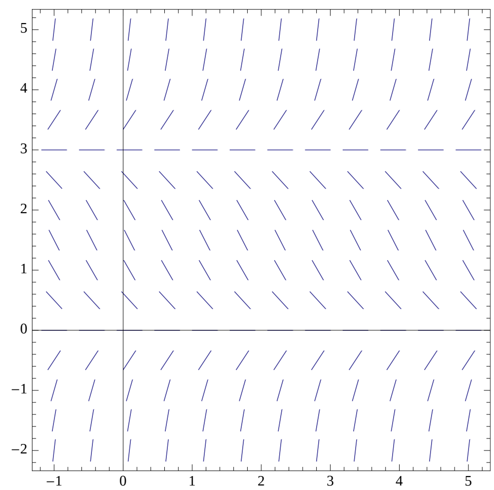
a. [5 points] Find the regions in the slope field of $y' = (y-3)y$ where the slopes are positive, negative or zero. Show all your computations.

Solution:
Positive : $y > 3$, $y < 0$
Zero: $y = 0, 3$
Negative: $0 < y < 3$

b. [2 points] Which of the following is the slope field of $y' = (y-3)y$? Circle your answer.



(a)



(b)

Solution: B)

c. [4 points] Find all the equilibrium solutions of $y' = (y-3)y$. Use the slope field of the equation to determine the stability of each equilibrium solution.

Solution:
$y = 3$ unstable
$y = 0$ stable

d. [4 points] Let $y(x)$ be the solution to the differential equation $y' = (y-x)y$ satisfying $y(1) = 2$. Use Euler's method with steps $\Delta x = \frac{1}{2}$ to approximate the value of $y(2)$. Show all your computations.

Solution: $(x_0, y_0) = (1, 2)$
$y_1 = 2 + (2-1)(2)(\frac{1}{2}) = 3$
$y_2 = 3 + (3 - \frac{3}{2})(3)(\frac{1}{2}) = 5.25$ then $y(2) \approx 5.25$.

5. [14 points] A particle moves on the unit circle according to the parametric equations

$$x(t) = -\sin(bt^2) \quad , \quad y(t) = \cos(bt^2) \qquad \text{and } b > 0.$$

for $0 \le t \le \pi$. Make sure to show all your work.

a. [1 point] What is the starting point of the particle?

Solution: $(x(0), y(0)) = (0, 1)$.

b. [2 points] In which direction (counterclockwise/clockwise) is the particle moving along the circle? Justify.

Solution: The particle moves around the unit circle making a **counterclockwise** angle $\theta = bt^2$ measured from the positive y axis. Since bt^2 is increasing, the particle never changes direction.

c. [5 points] Find an expression for the speed of the particle. Simplify it as much as possible.

Solution:
$x'(t) = -2tb\cos(bt^2) \qquad y'(t) = -2tb\sin(bt^2)$

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{(-2tb\cos(bt^2))^2 + (-2tb\sin(bt^2))^2} = 2tb$$

d. [2 points] At what value of t in $[0, \pi]$ is the speed of the particle the largest?

Solution: $v(t) = 2tb$ is the largest at $t = \pi$.

e. [4 points] Find the equation of the tangent line to the parametric equation at $t = \sqrt{\frac{\pi}{3b}}$.

Solution:
$x(\sqrt{\frac{\pi}{3b}}) = -\frac{\sqrt{3}}{2} = -.866, \qquad x'(\sqrt{\frac{\pi}{3b}}) = -2b\sqrt{\frac{\pi}{3b}}\left(\frac{1}{2}\right) = -\sqrt{\frac{\pi b}{3}}$

$y(\sqrt{\frac{\pi}{3b}}) = \frac{1}{2}, \qquad y'(\sqrt{\frac{\pi}{3b}}) = -2b\sqrt{\frac{\pi}{3b}}\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{\pi b}$

Parametric equation for the tangent line:
$x_{tan}(t) = -.866 - \sqrt{\frac{\pi b}{3}}t \qquad y_{tan}(t) = .5 - \sqrt{\pi b}t$

6. [15 points] Each of the integrals below are improper. Determine the convergence or divergence of each. Make sure you include all the appropriate steps to justify your answers. Approximations with your calculator will not receive credit.

a. [4 points]

$$\int_1^\infty \frac{5-2\sin x}{\sqrt{x^3}}dx$$

Solution: Since $-1 \le \sin x \le 1$ then $3 \le 5 - 2\sin x \le 7$. This yields

$$0 \le \frac{3}{\sqrt{x^3}} \le \frac{5-2\sin x}{\sqrt{x^3}} \le \frac{7}{\sqrt{x^3}}$$

$$0 \le \int_1^\infty \frac{5-2\sin x}{\sqrt{x^3}}dx \le \int_1^\infty \frac{7}{\sqrt{x^3}}dx \qquad \text{converges}$$

b. [5 points]

$$\int_1^2 \frac{x^2}{(x^3-1)^2}dx$$

Solution:

$$\int_1^2 \frac{x^2}{(x^3-1)^2}dx = \lim_{b\to 1^+}\int_b^2 \frac{x^2}{(x^3-1)^2}dx = \lim_{b\to 1^+} -\frac{1}{3(x^3-1)}\Big|_b^2$$
$$= \lim_{b\to 1^+}\frac{1}{3(b^3-1)} - \frac{1}{21} \qquad \text{diverges}$$

c. [6 points]

$$\int_2^\infty \frac{1}{(x^3+7)^{\frac{1}{3}}}dx$$

Solution:

$$\frac{1}{(x^3+7)^{\frac{1}{3}}} \ge \frac{1}{2x} \qquad \text{for } x \ge 1$$

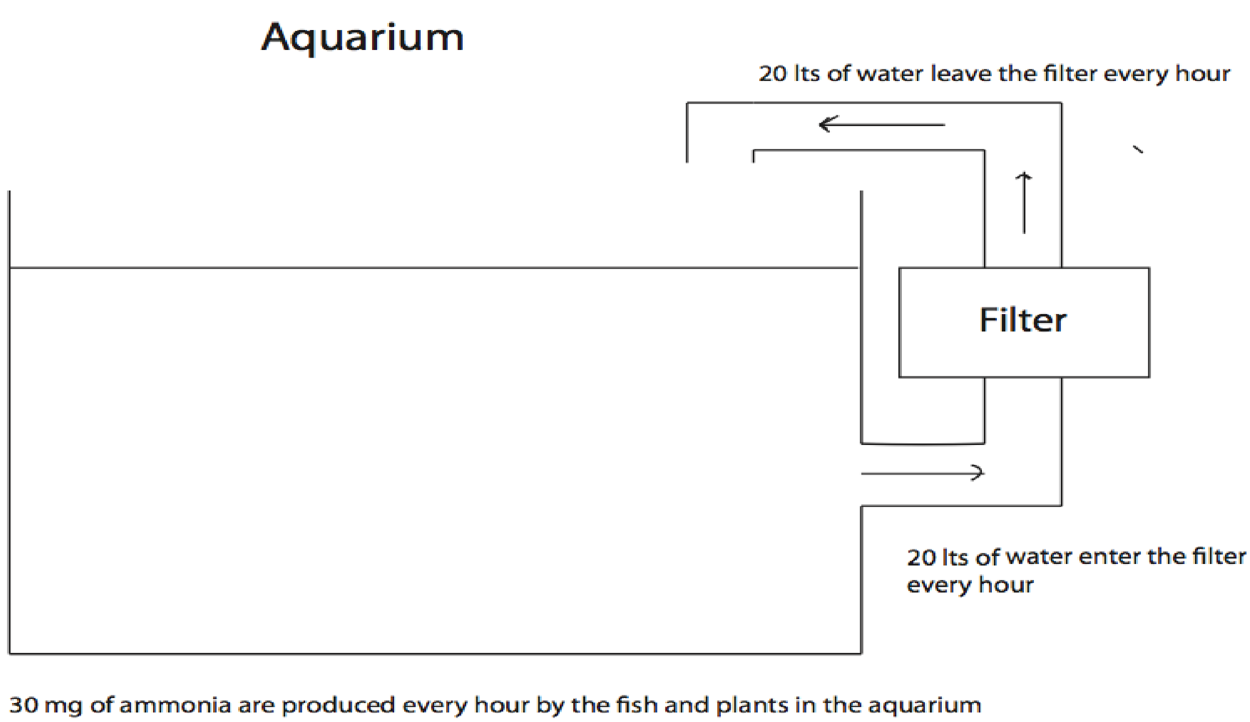
since for $x \ge 1$ we have $x^3 \ge 1$. Multiplying by 7 and adding x^3 we get $8x^3 \ge x^3 + 7$. Hence $\frac{1}{x^3+7} \ge \frac{1}{8x^3}$ and by taking the cube root on both sides we get $\frac{1}{(x^3+7)^{\frac{1}{3}}} \ge \frac{1}{2x}$. Hence

$$\int_2^\infty \frac{1}{(x^3+7)^{\frac{1}{3}}}dx \ge \int_2^\infty \frac{1}{2x}dx \qquad \text{diverges}$$

7. [15 points] An aquarium containing 100 liters of fresh water will be filled with a variety of small fish and aquatic plants.

A water filter is installed on the tank to help remove the ammonia produced by the decomposing organic matter generated by plants and fish in the aquarium. The filter takes water from the tank at a rate of 20 liters every hour. The water then is filtered and returned to the aquarium at the same rate of 20 liters every hour. Ninety percent of the ammonia contained in the water that goes through the filter is removed.

It is estimated that the fish and plants produce 30 mg of ammonia every hour. Assume the ammonia mixes instantly with the water in the aquarium.



a. [6 points] Let $Q(t)$ be the amount in mg of ammonia in the fish tank t hours after the fish were introduced into the aquarium. Find the differential equation satisfied by $Q(t)$. Include its initial condition.

Solution:

$$\frac{dQ}{dt} = 30 - 20\left(\frac{Q}{100}\right) + (.1)20\left(\frac{Q}{100}\right)$$
$$\frac{dQ}{dt} = 30 - .18Q \qquad Q(0) = 0$$

b. [7 points] Find the amount of ammonia in the fish tank 3 hours after the fish were introduced into the aquarium. Make sure to include units in your answer.

Solution:

$$\frac{dQ}{dt} = 30 - .18Q$$
$$\frac{dQ}{30 - .18Q} = dt$$
$$-\frac{1}{.18}\ln|30 - .18Q| = t + C$$
$$Q(t) = \frac{30}{.18} + De^{-.18t} = 166.6 + De^{-.18t}$$
$$\text{using } Q(0) = 0 \text{ you get} \qquad Q(t) = 166.6(1 - e^{-.18t})$$
$$Q(3) = 69.53 \text{ mg}$$

c. [2 points] What happens to the value of $Q(t)$ in the long run?

Solution:

$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 166.6(1 - e^{-.18t}) = 166.6.$

The value of $Q(t)$ converges to 166.6 mg.