1. [10 points] Indicate if each of the following statements are true or false by circling the correct answer. **Justify your answers.**

a. [2 points] If $F(x)$ is an antiderivative of an even function $f(x)$, then $F(x)$ must also be an even function.

True **[False]**

Solution: $f(x) = 3x^2$ has $F(x) = x^3 + 1$ as an antiderivative which is not even (not odd either).

b. [2 points] If $G(x)$ is an antiderivative of $g(x)$ and $(G(x) - F(x))' = 0$, then $F(x)$ is an antiderivative of $g(x)$.

[True] False

Solution: $g(x) = G'(x) = F'(x)$ hence $F(x)$ is an antiderivative of $g(x)$.

c. [2 points] Let $f(t) = bt + ct^2$ with $b > 0$ and $c > 0$, then $\text{Left}(n) \leq \int_0^{10} f(t)dt$ for all n.

[True] False

Solution: Since $f'(t) = b + 2ct > 0$ for $t > 0$, then $f(t)$ is increasing on $[0, 10]$ and the left sums yield an underestimate.

d. [2 points] The average of an even function $f(x)$ over the interval $[-a, a]$ is equal to twice its average over the interval $[0, a]$.

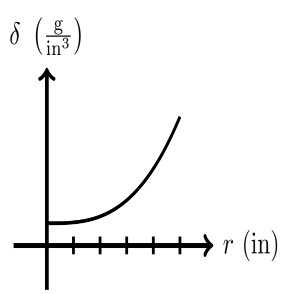
True **[False]**

Solution: Both average are the same. $\frac{1}{2a}\int_{-a}^{a} f(x)dx = \frac{2}{2a}\int_0^a f(x)dx = \frac{1}{a}\int_0^a f(x)dx$.

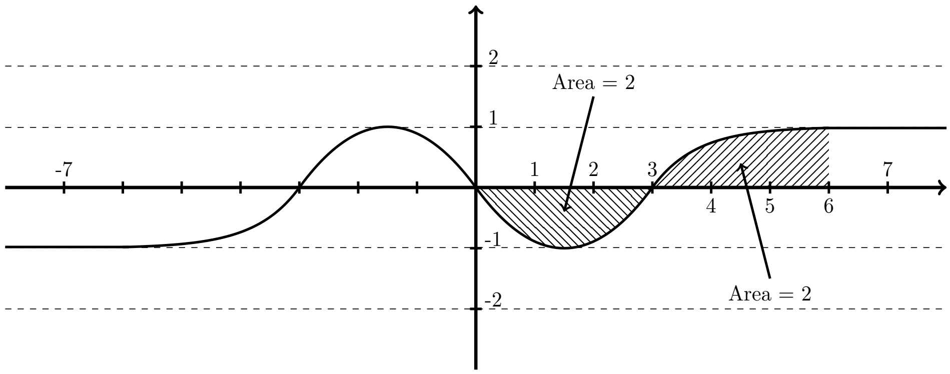
e. [2 points] The density δ of a circular porcelain dinner plate depends on the distance r from the center of the plate. The relationship between δ and r is shown in the graph below. The center of mass of this plate is located near the edge of the plate.

True **[False]**

Solution: The center of mass is at the center.



2. [17 points]
The graph of an odd function f is shown below.



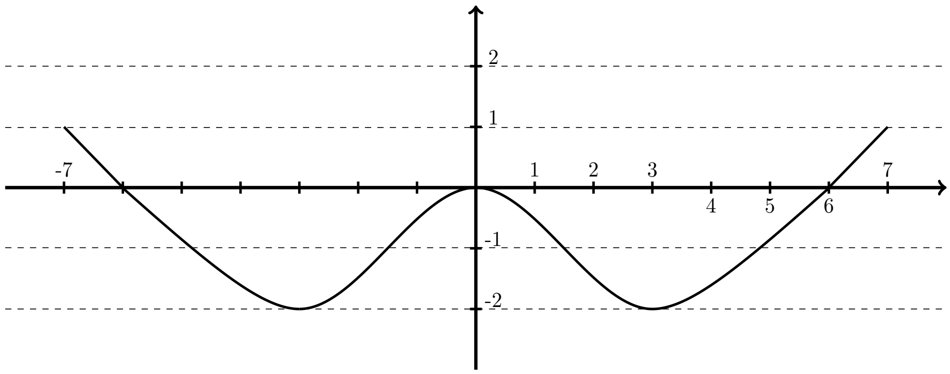
a. [7 points] Let $F(x)$ be the antiderivative of $f(x)$ with the property that $F(3) = -2$. Use the graph of $f(x)$ to compute the following values of $F(x)$.

x	-7	-6	-3	0	3	6	7
$F(x)$							

Solution:

x	-7	-6	-3	0	3	6	7
$F(x)$	1	0	-2	0	-2	0	1

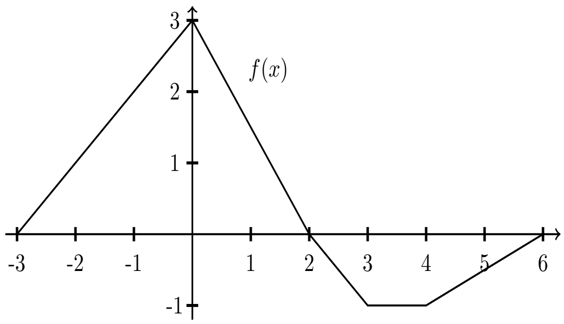
b. [8 points] Sketch the graph of $F(x)$ from $x = -7$ to $x = 7$. **Label all points of inflection**.

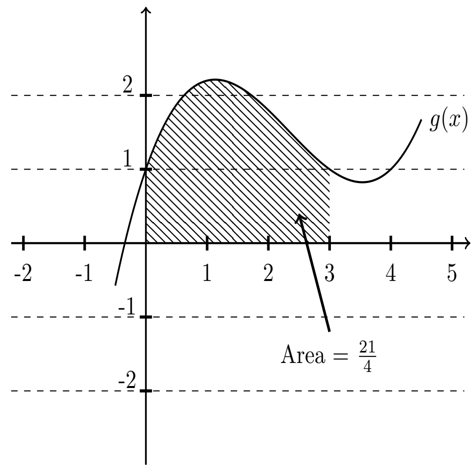


c. [2 points] Calculate the average value of f between $x = -3$ and $x = 7$.

Solution: $\frac{1}{7-(-3)} \int_{-3}^{7} f(x)dx = \frac{3}{10}$

3. [15 points] Use the graphs of $f(x)$ and $g(x)$ to find the EXACT values of A,B, and C. Show all your work.





a. [2 points] $A = \int_{-3}^{6} |f(x)| dx$

Solution: $A = \frac{15}{2} + \frac{5}{2} = 10$

b. [5 points] $B = \int_{0}^{2} xg'(x^2)dx$

Solution: Using $u = x^2$
$B = \frac{1}{2}\int_0^4 g'(u)du = \frac{1}{2}\left(g(4) - g(0)\right) = 0$

c. [8 points] $C = \int_0^3 2xg'(x)dx$

Solution: Using integration by parts $C = 2xg(x)|_0^3 - 2\int_0^3 g(x)dx = 6 - 2\left(\frac{21}{4}\right) = -\frac{9}{2}$

4. [15 points] A patient is given 100 mg of an experimental drug. It has been estimated that the rate $f(t)$ at which his body eliminates the drug is given in the following table. Values of t are in hours after the administration of the drug and $f(t)$ is measured in mg/hour.

t	0	0.5	1.0	1.5	2
$f(t)$	17.3	14.5	12.2	10.3	8.6

Assume $f(t)$ is continuous with no critical points or points of inflection in $0 \le t \le 2$. Make sure to include the appropriate units in your answers below.

a. [4 points] Use each left, right, trapezoid and midpoint sums to estimate amount of drug eliminated after 2 hours. When calculating each sum, use the maximum number of subdivisions possible. Show all the terms in each sum.

Solution:

$$\bullet Left(4) = .5(17.3 + 14.5 + 12.2 + 10.3) = 27.15 \text{ mg}$$
$$\bullet Right(4) = .5(14.5 + 12.2 + 10.3 + 8.6) = 22.8 \text{ mg}$$
$$\bullet Trap(4) = \frac{Left(4)+Right(4)}{2} = 24.97 \text{ mg}$$
$$\bullet Mid(2) = 1(14.5 + 10.3) = 24.8 \text{ mg.}$$

b. [4 points] Using the computations in a), what is the best overestimate you can find for the amount of drug removed from the patient's body after 2 hours? What is the best underestimate? Justify your answers.

Solution:

$$24.8 \le \int_0^2 f(t)dt \le 24.97.$$

$Trap(4)$ yields an overestimate and $Mid(2)$ is an underestimate since $f(t)$ is concave up.

c. [4 points] Using left and right hand sums, how often do we have to measure $f(t)$ in $0 \leq t \leq 2$ to obtain an estimate of the amount of drug eliminated from the patient's body after 2 hours within 0.1 mg of its actual value?

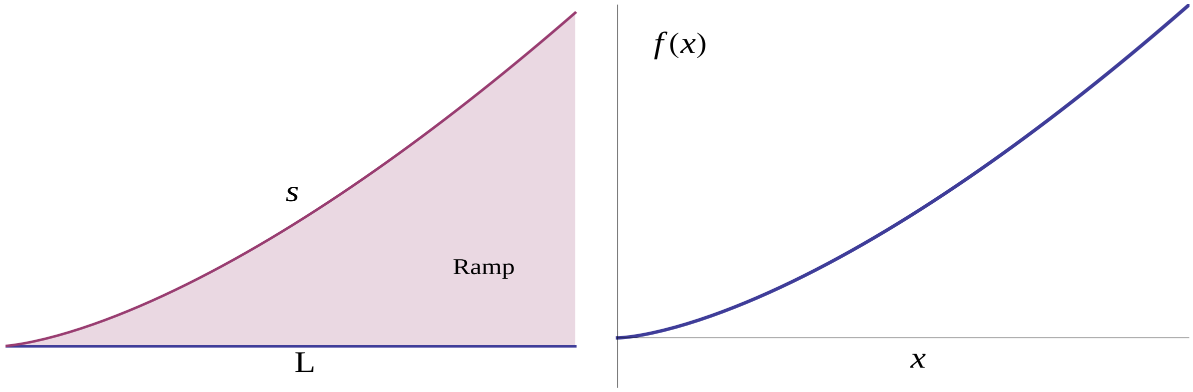
Solution: Since $f(t)$ is decreasing in $[0, 2]$, then we need to subdivide with pieces of length Δt satisfying $|17.3 - 8.6|\Delta t < 0.1$. Hence $\Delta t < .011$ hrs or more than 87 times every hour.

d. [3 points] Find a formula for $g(t)$, the amount of drug (in mg) left in the patient's body after t hours of being administered.

Solution:

$$g(t) = 100 - \int_0^t f(x)dx$$

5. [8 points] A company wants to design a bicycle ramp using the shape of the graph of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}}$,where x is the length in meters of the base of the ramp.

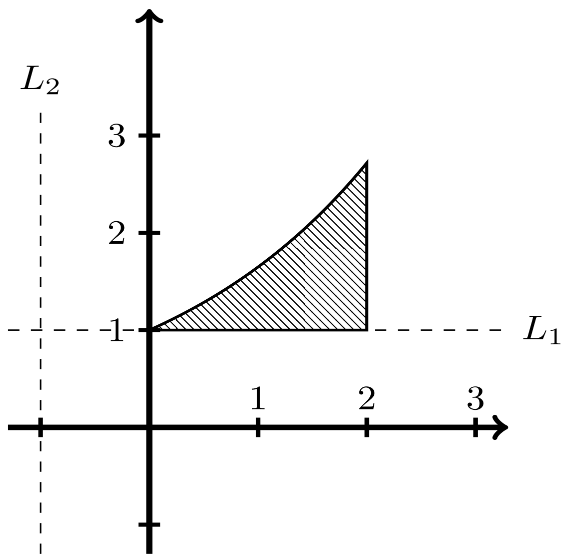


Find the length s of a ramp with base of length L. Show all your work.

Solution:

$$\begin{aligned} s &= \int_0^L \sqrt{1+(f'(x))^2}dx \\ &= \int_0^L \sqrt{1+(2\sqrt{x})^2}dx \\ &= \int_0^L \sqrt{1+4x}dx \\ &= \frac{1}{6}(1+4x)^{3/2}|_0^L \\ &= \frac{1}{6}(1+4L)^{3/2} - \frac{1}{6} \end{aligned}$$

6. [12 points]
The region bounded by the graph of $y = e^{0.5x}$, the line $y = 1$, and the line $x = 2$ is shown below. For each of the lines L_1 and L_2 write a definite integral that represents the volume of the solid object obtained by rotating the region around that line. You do not need to show your work or calculate the value of the integral.



a. [6 points] L_1:

Solution:

•Disks:

$$\int_0^2 \pi(e^{0.5x} - 1)^2 dx$$

•Washers: Intersection point $(2, e)$

$$\int_1^e 2\pi(y-1)(2 - 2\ln y)dy$$

b. [6 points] L_2:

Solution:

•Disks: Intersection point $(2, e)$

$$\int_1^e \pi((2+1)^2 - (2\ln(y) + 1)^2)dy$$

•Washers:

$$\int_0^2 2\pi(1+x)(e^{0.5x} - 1)dx$$

7. [11 points]
A truck carrying a large tank of paint leaves a garage at 9AM. The tank starts to leak in such a way that x miles from the garage, the density of paint on the road is $e^{-x^2/5000}$ gallons per mile. At 10AM, a cleaning crew leaves from the same garage and follows the path of the truck, scrubbing the paint from the road as it travels until it catches up to the leaking truck. At t hours after 10AM, the leaking truck is $50\ln(t+2)$ miles from the garage, and the cleanup crew is $35t$ miles from the garage. You may use your calculator to evaluate any definite integrals for this problem.

a. [4 points] Calculate the total amount of paint that has leaked from the truck by 11AM.

Solution: The truck at 11 AM is at $50\ln 3 = 54.9306$ miles from the garage. Total amount of paint leaked from the truck is

$$\int_0^{50\ln 3} e^{-x^2/5000}dx = 45.6246 \text{ gallons.}$$

b. [2 points] At time t hours after 10AM, what interval I of the road is still covered in paint? (you may assume that t represents a time before the trucks meet)

Solution: The truck is at $50\ln(t+2)$ miles from the garage and the crew is at $35t$ miles from the garage. $I = [35t, 50\ln(t+2)]$.

c. [3 points] Let $P(t)$ represent the amount of paint in gallons on the road t hours after 10 AM. Find a formula (which may include a definite integral) for $P(t)$.

Solution:

$$P(t) = \int_{35t}^{50\ln(t+2)} e^{-x^2/5000}dx$$

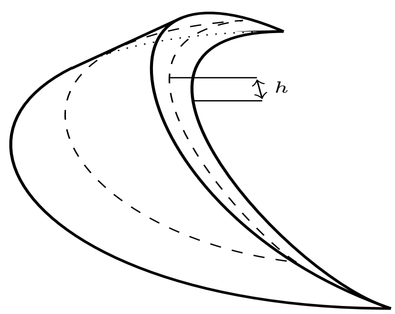
d. [2 points] Calculate $P'(1)$.

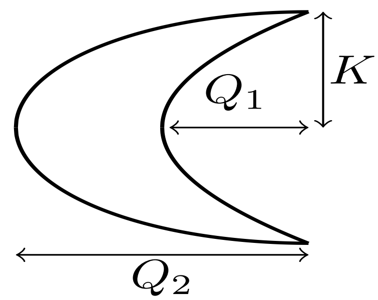
Solution:

$$P'(t) = e^{-(50\ln(t+2))^2/5000}\left(\frac{50}{t+2}\right) - 35\left(e^{-(35t)^2/5000}\right)$$

$$P'(1) = e^{-(50\ln(3))^2/5000}\left(\frac{50}{3}\right) - 35\left(e^{-(35)^2/5000}\right) = -18.2795 \text{ gal/hr}$$

8. [12 points] Sand dunes come in many shapes. *Barchan* dunes, which have the shape shown on the left, are studied extensively by geomorphologists. Horizontal cross-sections of these dunes are crescent-shaped (the dashed line encloses one such cross-section), and can be approximated as the shape on the right. The area of this shape is given by the formula $A_h = K(\frac{\pi}{2}Q_2 - \frac{4}{3}Q_1)$.





You are studying a barchan dune of 10 meters height, for which the values of Q_1, Q_2, and K vary with respect to the height h (in meters) of the cross-section according to the functions $Q_1(h) = 10 - h$, $Q_2(h) = 20 - 2h$, $K(h) = 100 - h^2$. The density of sand in the dune is $\delta = 1600$ kilograms per cubic meter.

a. [5 points] Write an expression for the volume of one slice of sand dune h meters above the ground and Δh meters thick.

Solution:

$$V_{slice} \approx A_h \Delta h = (100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right]\Delta h.$$

b. [5 points] Write a definite integral that represents the total mass of sand in the dune. You do not need to evaluate this integral.

Solution: Height of the dune = 10, so

$$M_{dune} = \int_0^{10} 1600(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh.$$

c. [2 points] Write an expression (involving integrals) for the height of the center of mass of the sand dune. You do not need to evaluate this integral.

Solution:

$$\bar{h} = \frac{\int_0^{10} 1600h(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}{\int_0^{10}(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}$$