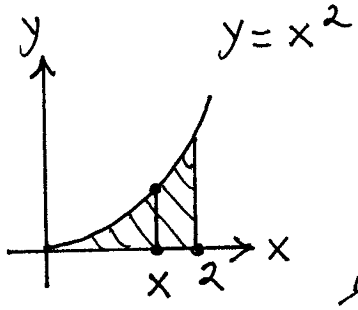
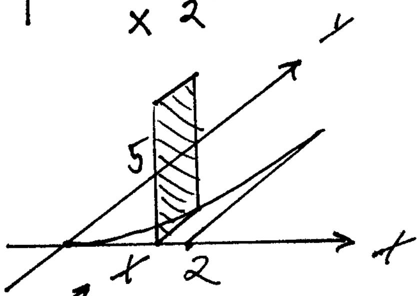
1.) (12 pts.) The base of a solid lies in the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$. Cross-sections of the solid taken perpendicular to the x-axis at x are rectangles of height 5. SET UP BUT DO NOT EVALUATE an integral which represents the volume of this solid.



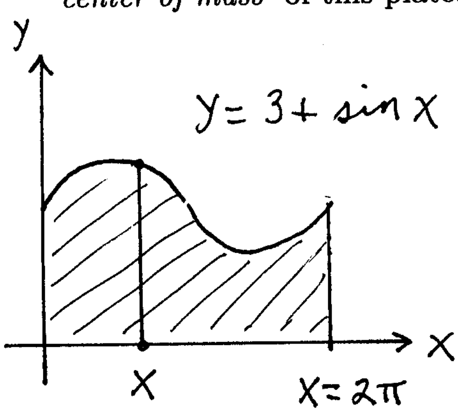


rectangle has area

$$A(x) = (\text{base})(\text{height})$$
$$= (x^2)(5) \;;$$

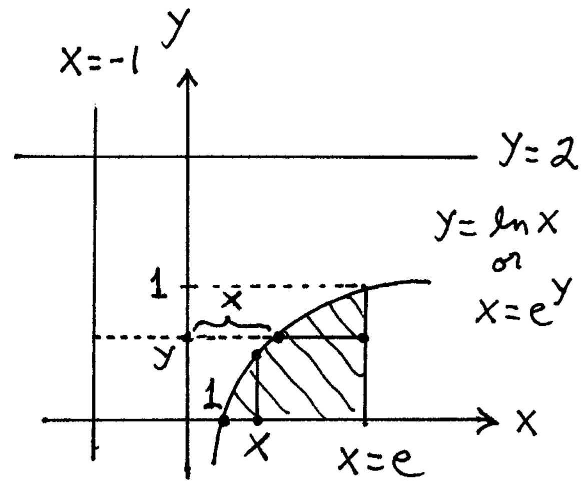
$$\text{Volume} = \int_0^2 A(x)\,dx$$
$$= \int_0^2 5x^2\,dx$$

2.) (12 pts.) A flat plate of *variable* density lies in the region bounded by the graphs of $y = 3 + \sin x$, $y = 0$, $x = 0$, and $x = 2\pi$. Density at point (x, y) is given by $\delta(x, y) = 3 + \sqrt{x}$. SET UP BUT DO NOT EVALUATE integrals which represent $\bar{y}$, the y-coordinate for the *center of mass* of this plate.



$$\bar{y} = \frac{\int_0^{2\pi} \frac{1}{2}(3+\sin x)^2 \cdot (3+\sqrt{x})\,dx}{\int_0^{2\pi} (3+\sin x)(3+\sqrt{x})\,dx}$$

3.) (6 pts. each) Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$. SET UP BUT DO NOT EVALUATE integrals which represent the volume of the solid formed by revolving this region about



a.) the x-axis using the DISC METHOD.

$$\text{Vol} = \pi \int_1^e \underbrace{(\ln x)}_{\text{radius}}^2 \, dx$$

b.) the y-axis using the SHELL METHOD.

$$\text{Vol} = 2\pi \int_1^e \underbrace{(x)}_{\text{radius}} \underbrace{(\ln x)}_{\text{height}} \, dx$$

c.) the line $x = -1$ using the DISC METHOD.

$$\text{Vol} = \pi \int_0^1 \underbrace{(e+1)}_{\text{outer radius}}^2 \, dy - \pi \int_0^1 \underbrace{(e^y+1)}_{\text{inner radius}}^2 \, dy$$

d.) the line $y = 2$ using the SHELL METHOD.

$$\text{Vol} = 2\pi \int_0^1 \underbrace{(2-y)}_{\text{radius}} \underbrace{(e-e^y)}_{\text{height}} \, dy$$

4.) (13 pts.) Set up and EVALUATE an integral which represents the *arc length* of the following curve, which is given parametrically by :

$$\begin{cases} x = (1/3)t^3 , \\ y = (1/2)t^2 , \quad \text{for } 0 \leq t \leq 1 . \end{cases}$$

$$\text{Arc} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(t^2)^2 + (t)^2}\, dt$$

$$= \int_0^1 \sqrt{t^4 + t^2}\, dt$$

$$= \int_0^1 \sqrt{t^2(t^2+1)}\, dt$$

$$= \int_0^1 t\sqrt{t^2+1}\, dt$$

$$= \frac{1}{\cancel{2}} \cdot \frac{(t^2+1)^{3/2}}{3/\cancel{2}} \Big|_0^1 = \frac{1}{3}\left(2^{3/2} - 1\right)$$

5.) (13 pts.) A chain weighs 2 pounds per foot and is used to raise a 100 pound object on the ground to a point 50 feet above the ground. Set up and EVALUATE an integral which represents the work required to complete the task (Include the weight of the chain in your solution.)

work required to move "system" from y to $y + dy$ is

$\approx$ (weight)(distance)

$$= \underbrace{(100}_{\text{object}} + \underbrace{(50-y)\cdot 2}_{\text{chain}}) \cdot dy$$

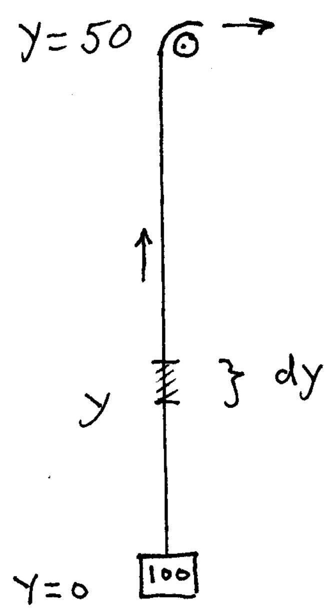
$= (200 - 2y)\, dy$ ft.-lbs. ;

so total work is

$$\text{Work} = \int_0^{50} (200 - 2y)\, dy$$

$$= (200y - y^2)\Big|_0^{50}$$

$$= 10{,}000 - 2500 = 7500 \text{ ft.-lbs.}$$



6.) (13 pts.) A flat plate is in the shape of a right triangle with legs 3 feet and 4 feet. It rests vertically on its 4-foot edge at the bottom of a pool filled to a depth of 8 feet. SET UP BUT DO NOT EVALUATE an integral which represents the force (on one side) of water pressure on this plate. (Water weighs 62.4 pounds per cubic foot.)

Force on thin strip is

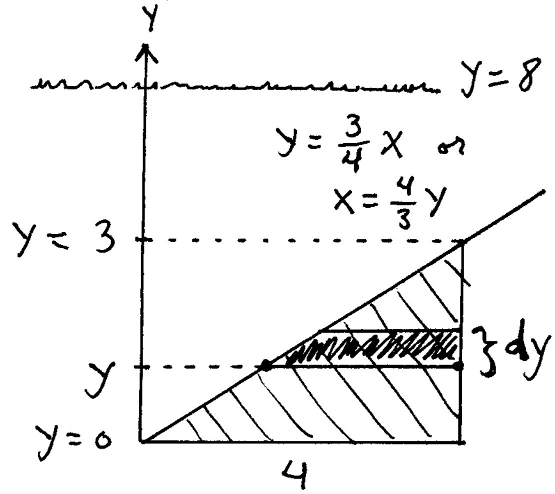
$\approx$ (area)(depth)(density)

$= \left(\left(4-\frac{4}{3}y\right)\cdot dy\right)\cdot(8-y)\cdot(62.4)$;

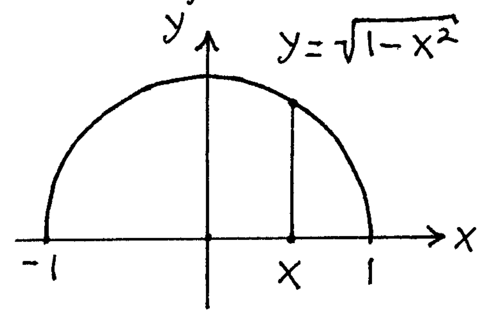
so total force on plate is

$$P = \int_0^3 (62.4)\cdot\left(4-\frac{4}{3}y\right)(8-y)\,dy$$

lbs.



7.) (13 pts.) The semi-circle $y=\sqrt{1-x^2}$ for $-1\le x\le 1$ is rotated about the x-axis to form a sphere. Set up and EVALUATE an integral which represents the *surface area* of this sphere.



$$\text{Area} = 2\pi\int_{-1}^{1} y\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\cdot\sqrt{1+\left(\tfrac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)\right)^2}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\cdot\sqrt{1+\frac{x^2}{1-x^2}}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\cdot\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx$$

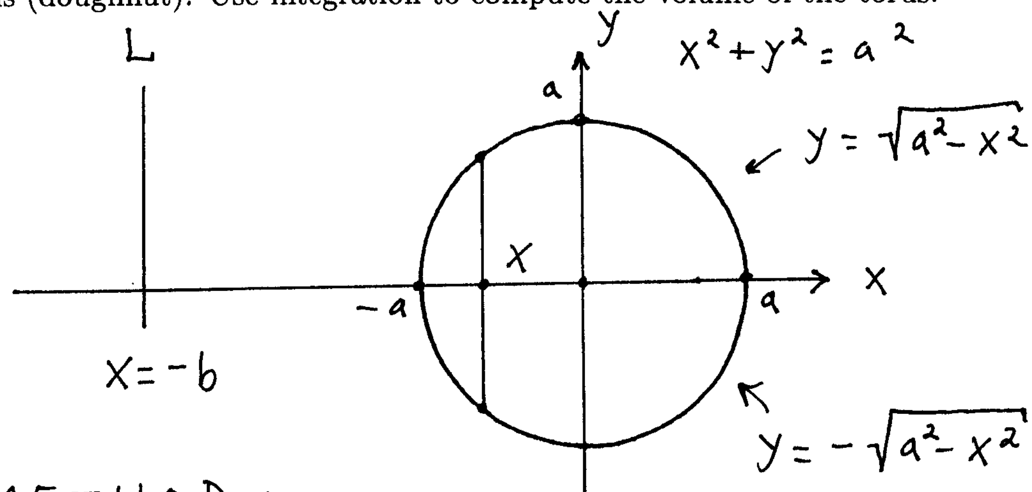
$$= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= 2\pi\int_{-1}^{1} 1\,dx$$

$$= 2\pi\cdot x\Big|_{-1}^{1} = 2\pi\cdot(1-(-1)) = 4\pi$$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) A circle has radius a units. Its center is b units from line L. The circle is rotated about line L to form a torus (doughnut). Use integration to compute the volume of the torus.



SHELL METHOD:

$$Vol = 2\pi \int_{-a}^{a} (x+b)\cdot 2\sqrt{a^2-x^2}\, dx$$

$$= 4\pi \int_{-a}^{a} x\sqrt{a^2-x^2}\, dx + 4\pi b \int_{-a}^{a} \sqrt{a^2-x^2}\, dx$$

$$= 4\pi \cdot \left(\frac{-1}{\cancel{2}}\right) \frac{(a^2-x^2)^{3/2}}{3/\cancel{2}} \Big|_{-a}^{a} + 4\pi b \cdot \frac{1}{2}\pi a^2$$

$$= \frac{-4}{3}\pi (0-0) + 2\pi b a^2$$

$$= 2\pi b a^2$$