Multiple Choice

(5 points each)

1. Calculate the average value of $f(x) = \frac{x}{x^2+1}$ over $0 \le x \le 2$.

(a) $\frac{1}{2}\tan^{-1} 2$ (b) $\frac{2}{5}$ (c) $\frac{1}{4}\ln 5$ (d) $\tan^{-1} 5$ (e) $\tan^{-1}(\ln 2)$

$$\bar{f} = \frac{1}{2-0}\int_0^2 \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\int \frac{x\,dx}{x^2+1}\right]_{x=0}^{x=2}$$

Let $u = x^2+1$. Then $du = 2x\,dx$, so

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C.$$

Therefore,

$$\bar{f} = \frac{1}{4}\ln(x^2+1)\Big|_{x=0}^{x=2}$$

$$= \frac{1}{4}\ln 5 - \frac{1}{4}\ln 1 = \frac{1}{4}\ln 5$$

2. Suppose the force needed to displace an object depends on the position x of the object according to the function

$$F(x) = 3x^2 - 4x + 6.$$

If the object is displaced from $x = 1$ to $x = 2$, calculate the work that is done.

(a) 6 (b) 5 (c) 7 (d) 10 (e) 12

$$W = \int_1^2 (3x^2 - 4x + 6)\,dx = \left[x^3 - 2x^2 + 6x\right]_{x=1}^{x=2}$$

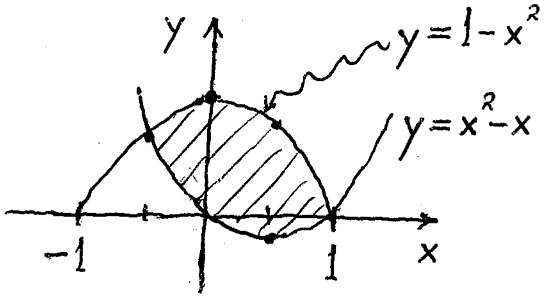
$$= [8 - 8 + 12] - [1 - 2 + 6]$$

$$= 12 - 5 = 7$$

3. Which integral is the area enclosed by the curves $y = x^2 - x$ and $y = 1 - x^2$?

(a) $\int_{-\frac{1}{2}}^{1} (2x^2 - x - 1)dx$ (b) $\int_{-\frac{1}{4}}^{1} (-2x^2 + x + 1)dx$ (c) $\int_{-\frac{1}{4}}^{\frac{3}{4}} (1 - y)dy$

(d) $\int_{-\frac{1}{2}}^{1} (-2x^2 + x + 1)dx$ (e) $\int_{0}^{1} (-2x^2 + x + 1)dx$



$$x^2 - x = 1 - x^2$$
$$2x^2 - x - 1 = 0$$
$$(2x+1)(x-1) = 0$$
$$x = -\tfrac{1}{2}, 1$$

$$A = \int_{-\frac{1}{2}}^{1} [(1-x^2) - (x^2 - x)]dx$$
$$= \int_{-\frac{1}{2}}^{1} (-2x^2 + x + 1)dx$$

4. Calculate the indefinite integral $\int x \cos x \, dx$.

(a) $\frac{1}{2}x^2 \sin x + C$ (b) $x \sin x + \cos x + C$ (c) $\cos x - x \sin x + C$ (d) $-2x \sin x + C$

(e) $x \sin^2 x + C$

Actually, one can differentiate each choice and see which one yields the integrand. Indeed,

$$\frac{d}{dx}(x \sin x + \cos x) = (\sin x + x \cos x) + (-\sin x)$$
$$= x \cos x$$

5. Calculate the indefinite integral $\int x^5\sqrt{x^3+1}\,dx$.

(a) $\frac{1}{6}x^6\sqrt{\frac{1}{4}x^4+x+C}$

(b) $\frac{2}{15}(x^3+1)^{\frac{5}{2}}-\frac{2}{9}(x^3+1)^{\frac{3}{2}}+C$

(c) $5x^4\sqrt{x^3+1}+\frac{3}{2}x^7(x^3+1)^{-\frac{1}{2}}+C$

(d) $\frac{2}{15}x^{\frac{15}{2}}+\frac{1}{6}x^6+C$

(e) $\frac{6}{19}(x^3+1)^{\frac{19}{6}}-\frac{2}{3}(x^3+1)^{\frac{3}{2}}+C$

Again, one can differentiate each choice and see which one yields the integrand. However, if this is too time-consuming, one can integrate by substitution. Let $u=x^3+1$. Then $du=3x^2dx$ and $x^3=u-1$. Hence

$$\int x^5\sqrt{x^3+1}\,dx=\frac{1}{3}\int(u-1)\sqrt{u}\,du=\frac{1}{3}\int u^{3/2}du-\frac{1}{3}\int u^{1/2}du$$

$$=\frac{2}{15}u^{5/2}-\frac{2}{9}u^{3/2}+C=\frac{2}{15}(x^3+1)^{5/2}-\frac{2}{9}(x^3+1)^{3/2}+C$$

6. Which integral is the volume of the solid generated by revolving about the y-axis the region enclosed by the curves $y=1-x^2$ and $y=(x-1)^2$?

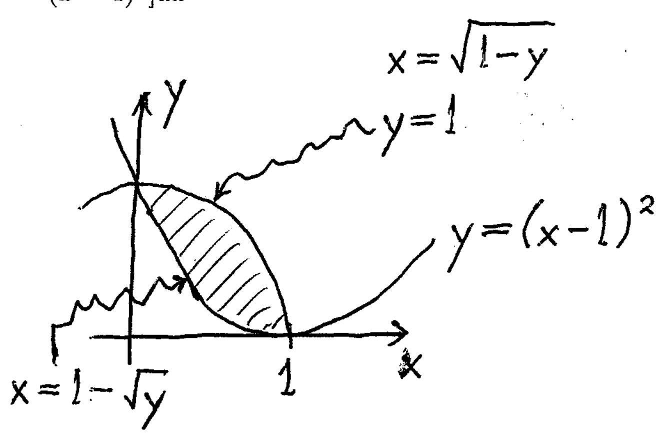
(a) $2\pi\int_0^1 x(2x-2x^2)dx$

(b) $\pi\int_0^1[(1-x^2)^2-(x-1)^4]dx$

(c) $2\pi\int_0^1 y(\sqrt{1-y}-\sqrt{y}+1)dy$

(d) $\int_0^1[1-x^2-(x-1)^2]dx$

(e) $\int_0^1(\sqrt{1-y}-\sqrt{y}+1)dy$



If we choose to integrate with respect to x with the region revolved about the y-axis, then the correct formula is the shell formula:

$$V=2\pi\int_0^1 x[(1-x^2)-(x-1)^2]dx.$$

For integration with respect to y, the correct formula is the washer formula:

$$V=\pi\int_0^1[(1-y)-(1-\sqrt{y})^2]dy.$$

7. Compute $\int_1^e \sqrt{x} \ln x \, dx$.

(a) 1 (b) $3\sqrt{e}-3$ (c) $\frac{2}{3}e^{\frac{3}{2}}(e-1)$ (d) $\frac{2}{9}e^{\frac{3}{2}}+\frac{4}{9}$ (e) $\frac{3}{2}e^{-\frac{1}{2}}-1$

$$\int_1^e \sqrt{x} \ln x \, dx = \left[\int \sqrt{x} \ln x \, dx \right]_{x=1}^{x=e}$$

Let $u = \ln x$ and $v = \frac{2}{3}x^{3/2}$. Then $dv = x^{1/2}dx$, and so

$$\int \sqrt{x} \ln x \, dx = \int u \, dv = uv - \int v \, du.$$

Since $du = \frac{dx}{x}$, it follows that

$$\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{3/2}\frac{1}{x}dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.$$

Thus $\int_1^e \sqrt{x} \ln x \, dx = \left[\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}\right]_{x=1}^{x=e} = \left[\frac{2}{3}e^{3/2} - \frac{4}{9}e^{3/2}\right] - \left[-\frac{4}{9}\right]$

8. Compute $\int_0^\pi \sin^3 x \cos^2 x \, dx$.

(a) 0 (b) $\frac{4}{15}$ (c) $\frac{2}{3}\pi$ (d) 1 (e) $\frac{5}{4}$

Let $u = -\cos x$. Then $du = \sin x \, dx$ and $\sin^2 x = 1 - u^2$. Thus

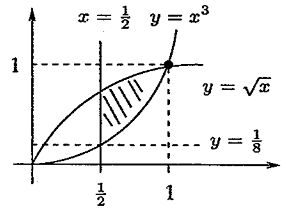
$$\int_0^\pi \sin^3 x \cos^2 x \, dx = \int_{-1}^1 (1-u^2)u^2 du$$

$$= \int_{-1}^1 (u^2 - u^4) du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_{u=-1}^{u=1}$$

$$= \left[\frac{1}{3} - \frac{1}{5}\right] - \left[-\frac{1}{3} + \frac{1}{5}\right]$$

$$= \frac{4}{15}.$$



9. Consider the solid generated by revolving about the line $y=\dfrac{1}{8}$ the region bounded by $y=\sqrt{x}$, $y=x^3$, and $x=\dfrac{1}{2}$. Which integral is the volume?

(a) $2\pi\int_{\frac{1}{2}}^{1}\left(x-\frac{1}{2}\right)(\sqrt{x}-x^3)dx$ (b) $2\pi\int_{\frac{1}{8}}^{1}\left(y-\frac{1}{8}\right)(\sqrt[3]{y}-1)dy$

(c) $\pi\int_{\frac{1}{8}}^{1}\left[\left(\sqrt[3]{y}-\frac{1}{2}\right)^2-\left(y^2-\frac{1}{2}\right)^2\right]dy$ (d) $2\pi\int_{\frac{1}{8}}^{1}\left(y-\frac{1}{8}\right)(\sqrt[3]{y}-y^2)dy$

(e) $\pi\int_{\frac{1}{2}}^{1}\left[\left(\sqrt{x}-\frac{1}{8}\right)^2-\left(x^3-\frac{1}{8}\right)^2\right]dx$

Integration with respect to x is the easiest way to handle this one. This choice calls for the washer formula, where the inner and outer radii are distances from $y=\frac{1}{8}$. Thus

$$V=\pi\int_{\frac{1}{2}}^{1}\left[\left(\sqrt{x}-\tfrac{1}{8}\right)^2-\left(x^3-\tfrac{1}{8}\right)^2\right]dx.$$

10. A boy scout scales a 12-foot rock face while attached by a rope to an 84-pound pack he has left at the base of the rock face. When he reaches the top, how much work is done by hauling the pack up after him if the rope itself weighs $\frac{1}{3}$ of a pound per foot? (Assume that at any given stage, the amount of rope that has been drawn up does not contribute to the weight anymore.)

(a) 1032 ft. lb. (b) 1008 ft. lb. (c) 1080 ft. lb. (d) 504 ft. lb. (e) 540 ft. lb.

If x is the length of rope still hanging out at a given stage, then the total weight – and therefore the force applied – is given by

$$F(x)=\tfrac{1}{3}x+84.$$

Thus

$$W=\int_0^{12}F(x)\,dx=\left[\tfrac{1}{6}x^2+84x\right]_{x=0}^{x=12}$$

$$=\tfrac{1}{6}(12)^2+84(12)=1032.$$

Work-Out Problems

Part 2

Show your work. No credit will be given to unsupported answers.

11. Calculate the following integrals.

(a) $\int_1^2 \frac{\ln x}{x}\,dx$ (6 points)

Let $u = \ln x$. Then $du = \frac{dx}{x}$, and so

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du = \left[\tfrac{1}{2}u^2\right]_{u=0}^{u=\ln 2} = \tfrac{1}{2}(\ln 2)^2.$$

(b) $\int x^2 e^x\,dx$ (7 points)

Let $u = x^2$ and $v = e^x$. Then $dv = e^x dx$, and so

$\int x^2 e^x dx = \int u\,dv = uv - \int v\,du.$

Since $du = 2x\,dx$, this means $\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx.$

Let $\tilde{u} = x$. Then $\int x e^x dx = \int \tilde{u}\,dv = \tilde{u}v - \int v\,d\tilde{u}.$

Since $d\tilde{u} = dx$, this means $\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$

Thus $\int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x + C) = x^2 e^x - 2xe^x + 2e^x + C.$

(c) $\int \tan^4 x\,dx$ (7 points)

Since $\sec^2 x - 1 = \tan^2 x$, we may write

$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x.$

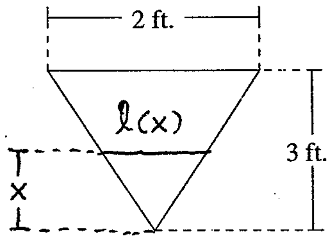
$\int \tan^2 x \sec^2 x\,dx = \int u^2 du$ with $u = \tan x$ (because $du = \sec^2 x\,dx$)

Thus $\int \tan^2 x \sec^2 x\,dx = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C.$

Moreover, $\int \tan^2 x\,dx = \int \sec^2 x\,dx - \int dx = \tan x - x + C.$

Therefore, $\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$

12. Consider a trough full of liquid with weight density ρg, where the trough is 8 ft. long and its cross-section is given by the figure



Calculate the work needed to pump all of the fluid over the top. (10 points)

$W = \rho g \int_0^3 (3-x) A(x)\,dx$, where $A(x)$ is the horizontal area at level x.

$A(x) = 8\ell(x)$, while $\frac{\ell(x)}{x} = \frac{2}{3}$ by similar triangles.

Thus $A(x) = \frac{16}{3}x$.

$$W = \rho g \int_0^3 \frac{16}{3}x(3-x)\,dx = \rho \int_0^3 \left(16x - \frac{16}{3}x^2\right)dx$$

$$= \rho g \left[8x^2 - \frac{16}{9}x^3\right]_{x=0}^{x=3} = \rho g\,[72-48] = 24\rho g.$$

13. Consider the curves $y = x^2+1$ and $y = x+3$. Find the area between these curves from $x=0$ to $x=3$. (10 points)

$x^2+1 \leq x+3$ for $0 \leq x \leq 2$ (because $(x^2+1)-(x+3) = x^2-x-2$
$= (x-2)(x+1) \leq 0$ on that region)

$x^2+1 \geq x+3$ for $2 \leq x \leq 3$ (because $(x-2)(x+1) \geq 0$ on that region)

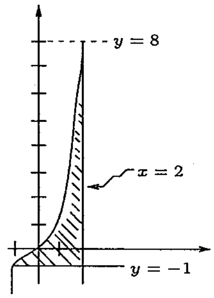
$$A = \int_0^2 (x+3-x^2-1)\,dx + \int_2^3 (x^2+1-x-3)\,dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{x=0}^{x=2} + \left[\frac{1}{3}x^3 - 2x - \frac{1}{2}x^2\right]_{x=2}^{x=3}$$

$$= \left[\frac{10}{3} - 0\right] + \left[-\frac{3}{2} - \left(-\frac{10}{3}\right)\right] = \frac{31}{6}.$$

9

14. Consider a solid with a base in the xy-plane given as the region bounded by $y = x^3$, $x = 2$, and $y = -1$. What is the volume of the solid if every cross-section perpendicular to the x-axis is a square? (10 points)



The interval is $-1 \leq x \leq 2$.

$V = \int_{-1}^{2} A(x)\,dx$, where $A(x)$ is the area of the cross-section.

$A(x) = \ell(x)^2$, where $\ell(x)$ is the length of the side.

$\ell(x) = x^3 + 1$

$$V = \int_{-1}^{2} (x^3+1)^2 dx = \int_{-1}^{2} (x^6 + 2x^3 + 1)dx$$

$$= \left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_{x=-1}^{x=2}$$

$$= \left[\frac{128}{7} + 8 + 2\right] - \left[-\frac{1}{7} + \frac{1}{2} - 1\right]$$

$$= \frac{129}{7} + \frac{21}{2} = \frac{405}{14}$$