Problem 1. Let $f(x) = \dfrac{e^x + e^{-x}}{2}$.

(a) Find the average value of f on the interval $[-1, 1]$.

Answer:

The average value of f on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

$$\frac{1}{2}\int_{-1}^{1} \frac{e^x + e^{-x}}{2}\,dx = \frac{1}{2}\left(\frac{e^x - e^{-x}}{2}\right)\Bigg]_{-1}^{1} = \frac{1}{2}\left(e - \frac{1}{e}\right) = 1.752\ldots$$

(b) Find the length of $y = f(x)$ from $x = -1$ to $x = 1$.

Answer:

$$\begin{aligned}
L = \int_{-1}^{1} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\,dx &= \int_{-1}^{1} \sqrt{\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1}\,dx \\
&= \int_{-1}^{1} \sqrt{\frac{e^{2x} - 2 + e^{-2x}}{4} + 1}\,dx \\
&= \int_{-1}^{1} \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\,dx \\
&= \int_{-1}^{1} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\,dx \\
&= \int_{-1}^{1} \frac{e^x + e^{-x}}{2}\,dx \\
&= \frac{e^x - e^{-x}}{2}\Bigg]_{-1}^{1} \\
&= e - \frac{1}{e} = 2.350\ldots
\end{aligned}$$

Problem 2. Solve the differential equation

$$y' = \frac{y \cos x}{1 + y^2}$$

subject to the initial condition $y(0) = 1$.

Answer:

$$\frac{dy}{dx} = \frac{y\cos(x)}{1+y^2} \Rightarrow \frac{1+y^2}{y}dy = \cos(x)dx \Rightarrow \int \frac{1}{y} + y\,dy = \int \cos(x)\,dx \Rightarrow \ln(y) + \frac{1}{2}y^2 = \sin(x) + C$$

The condition that $y(0) = 1$ means that C satisfies

$$\ln(1) + \frac{1}{2}1^2 = \sin(0) + C \Rightarrow \frac{1}{2} = C.$$

So, we have y satisfies the equation

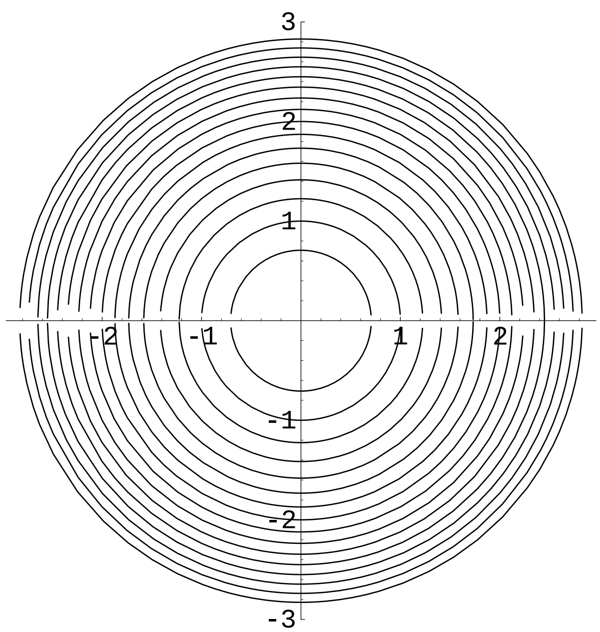
$$\ln(y) + \frac{1}{2}y^2 = \sin(x) + \frac{1}{2}.$$

Problem 3. Sketch the solution curves to the differential equation $y' = -\dfrac{x}{y}$.

Answer:
We have

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C \Rightarrow x^2 + y^2 = 2C.$$

So, the solution curves are all circles centered at the origin.



Problem 4. True or False?

(a) If y is any solution to $y' = x^4 + y^4 + y^2(1-x)$, then y is increasing when $x < 1$.

Answer:

True, since $y' > 0$ when $x < 1$.

(b) If W is the solution to the differential equation $\frac{dW}{dt} = \frac{3}{1000}W$ satisfying the initial condition $W(0) = 100$, then $W\left(\frac{1000}{3}\ln(5)\right) = 500$.

Answer:

True. You can see that from $\frac{dW}{dt} = \frac{3}{1000}W$ and $W(0) = 100 \Rightarrow W(t) = 100e^{\frac{3}{1000}t}$.

(c) If W is the solution to the differential equation $\frac{dW}{dt} = \frac{3}{1000}W\left(1 - \frac{W}{400}\right)$ satisfying the initial condition $W(0) = 100$, then $\lim_{t\to\infty} W(t) = 400$.

Answer:

True. Here W satisfies the logistic equation with a max population of 400.

(d) If C is onc solution curve for the differential equation $y' = \frac{x^3 + 2x - e^x}{1 + e^x}$, then infinitely many other solution curves can be obtained by shifting C to the right or left.

Answer:

False. Here, y' depends on x so the solution curves through (x_1, y) and (x_2, y) for different x_1 and x_2 will definitely have a different shape. Note, however, that y' doesn't depend on x here, so the solutions will have the form $y = f(x) + K$ and all solutions will be obtained by *vertical* translation of any one particular solution curve.

(e) If C is onc solution curve for the differential equation $y' = \frac{y^3 + 2y - e^y}{1 + e^y}$, then infinitely many other solution curves can be obtained by shifting C to the right or left.

Answer:

True. This is because y' does not depend on x.

Problem 5. Consider the system of differential equations governing the population of aphids and ladybugs:

$$\frac{dA}{dt} = 2A\left(1 - \frac{1}{10000}A\right) - \frac{1}{100}AL$$
$$\frac{dL}{dt} = -\frac{1}{2}L + \frac{1}{10000}AL$$

(a) Describe what will happen to the population of aphids in the absence of ladybugs.

Answer:

(Note that this prob

Here, in the absence of ladybugs, A satisfies the logistic equation $\frac{dA}{dt} = 2A\left(1 - \frac{1}{10000}A\right)$ and one can see that $\lim_{t\to\infty} A(t) = 10000$.

(b) Find the equilibrium solution of the system.

Answer:

There are three solutions to the equations

$$0 = 2A\left(1 - \frac{1}{10000}A\right) - \frac{1}{100}AL$$
$$0 = -\frac{1}{2}L + \frac{1}{10000}AL$$

We have $(A, L) = (0, 0)$, $(A, L) = (1000, 0)$, and $(A, L) = 5000, 100)$.

(c) Find an expression for $\dfrac{dL}{dA}$.

Answer:

From the chain rule: $\dfrac{dL}{dA} = \dfrac{\frac{dL}{dt}}{\frac{dA}{dt}}$. So,

$$\frac{dL}{dA} = \frac{-\frac{1}{2}L + \frac{1}{10000}AL}{2A\left(1 - \frac{1}{10000}A\right) - \frac{1}{100}AL}.$$

Problem 6. Use Euler's method with step size 0.2 to approximate $y(0.4)$ where $y(x)$ is the solution to the differential equation $y' = 2xy^2$ with the initial value $y(0) = 1$.

Answer:
Here, we have

$$(x_0, y_0) = (0, 1) \Rightarrow y_0' = 2x_0y_0^2 = 0 \Rightarrow y_1 = y_0 + (0.2)y_0' = 1 + 0 = 1$$
$$(x_1, y_1) = (0.2, 1) \Rightarrow y_0' = 2x_1y_1^2 = 0.4 \Rightarrow y_2 = y_1 + (0.2)y_1' = 1 + (0.2)(0.4) = 1.08$$
$$(x_2, y_2) = (0.4, 1.08).$$

Problem 7. Find a formula for the general term a_n of the sequence assuming that the general pattern of the first few terms continues:

(a) $\{1, 6, 11, 16, \ldots\}$

Answer:

$a_n = 5n - 4$

(b) $\left\{-\frac{3}{4}, \frac{9}{16}, -\frac{27}{64}, \ldots\right\}$

Answer:

$a_n \left(-\frac{3}{4}\right)^n$

Problem 8. Does the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converge or diverge? Explain.

Answer:
This series $\sum_{n=1}^{\infty} \frac{n}{2n+1} = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \cdots$ diverges. Note that

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0.$$

Therefore, by the n-th term test, the series diverges.

Problem 9. Find the constant A so that

$$\sum_{k=2}^{\infty} A\left(\frac{2}{9}\right)^k = 5.$$

Answer:
This is a geometric series with ratio $r = \frac{2}{9}$. Thus, the series converges (since $|\frac{2}{9}| < 1$) and has the sum

$$\sum_{k=2}^{\infty} A\left(\frac{2}{9}\right)^k = \frac{A\left(\frac{2}{9}\right)^2}{1-\frac{2}{9}} = \frac{\frac{4}{81}A}{\frac{7}{9}} = \frac{4}{63}A.$$

So, if

$$\sum_{k=2}^{\infty} A\left(\frac{2}{9}\right)^k = 5,$$

then we have

$$\frac{4}{63}A = 5 \Rightarrow A = \frac{315}{4}.$$