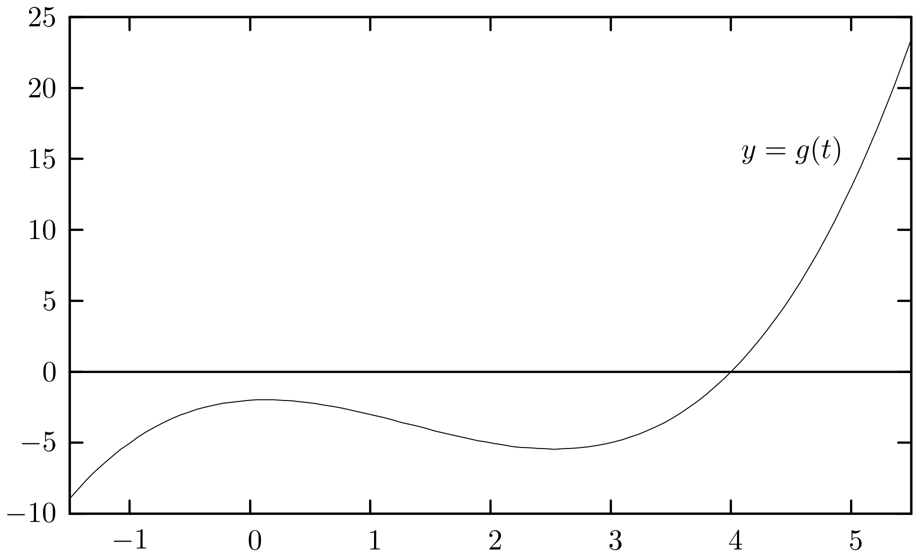
Problem 1. The picture below shows the graph of a function g.



(a) **[10 points]** Find $\int_{-1}^{5} g'(t)\,dt$.

Answer:

By the fundamental theorem of calculus (the "evalutation theorem" in the text), we have:

$$\int_{-1}^{5} g'(t)\,dt = g(5) - g(-1) = 15 - (-5) = 20.$$

(b) **[10 points]** Let $A(x) = \int_{1}^{x} g(t)\,dt$. Find $A'(2)$.

Answer:

Here, we use the other part of the fundamental theorem of calculus:

$$A'(x) = \frac{d}{dx}\left(\int_{1}^{x} g(t)\,dt\right) = g(x) \text{ and so } A'(2) = g(2) = -5.$$

Problem 2. [**20 points**] Determine whether $\int_1^\infty \frac{\sin^2(x)}{x^3}\,dx$ converges or diverges. Justify your answer completely.

Answer:
We use the comparison theorem to show that

$$\int_1^\infty \frac{\sin^2(x)}{x^3}\,dx \text{ converges.}$$

First, observe that $\int_1^\infty \frac{dx}{x^3}$ converges:

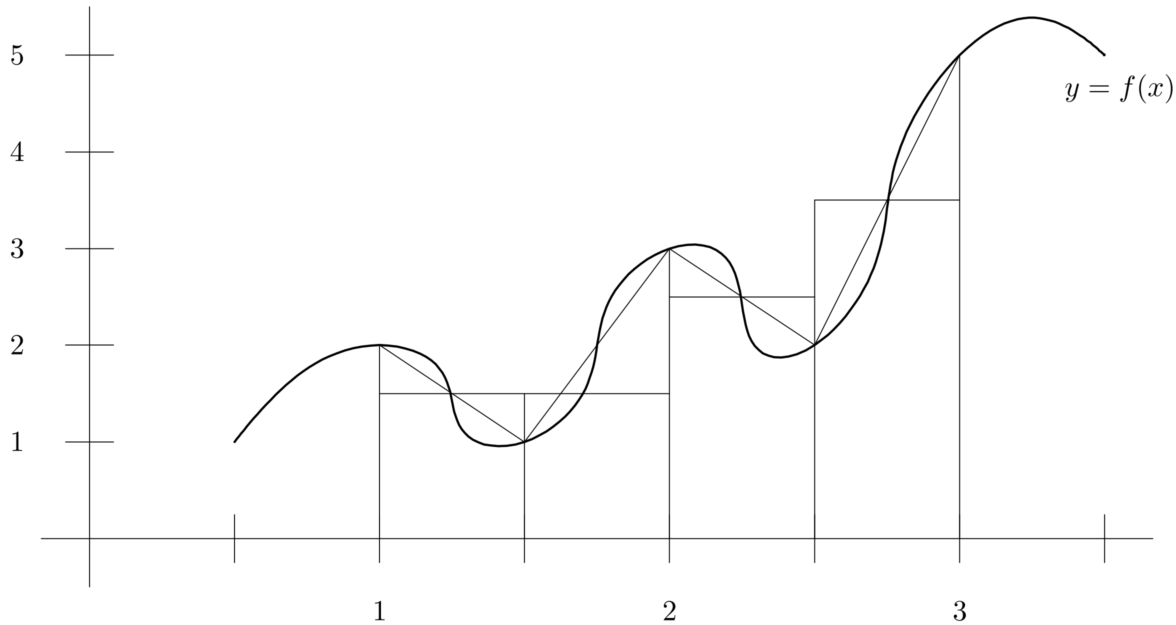
$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to\infty} -\frac{1}{2x^2}\Big]_1^b = \lim_{b\to\infty} -\frac{1}{2b^2} - \left(-\frac{1}{(2)(1)^2}\right) = \frac{1}{2}.$$

Now, since $-1 \le \sin(x) \le 1$ for any x, we have

$$0 < \frac{\sin^2(x)}{x^3} < \frac{1}{x^3}.$$

This inequality, together with the fact that $\int_1^\infty \frac{dx}{x^3}$ converges, implies that $\int_1^\infty \frac{\sin^2(x)}{x^3}\,dx$ converges by the comparison theorem.

Problem 3. Below is a sketch of $y = f(x)$. The polygonal paths may make it easier to approximate $\int_1^3 f(x)dx$.



(a) **[10 points]** Use the trapezoid rule with $n = 4$ to approximate $\int_1^3 f(x)dx$.

Answer:

The trapezoid rule, with $n = 4$ gives:

$$\int_1^3 f(x)dx \approx \frac{\Delta x}{2}\left(f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right)$$
$$= \frac{1}{4}\left(2 + 2(1) + 2(3) + 2(2) + 5\right) = \frac{19}{4}.$$

(b) **[10 points]** Use the midpoint rule with $n = 4$ to approximate $\int_1^3 f(x)dx$.

Answer:

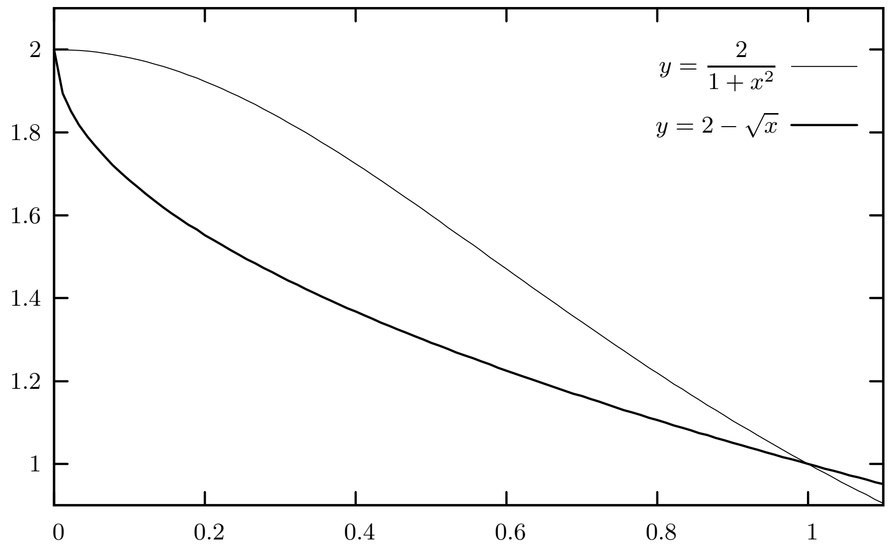
The midpoint rule, with $n = 4$ gives:

$$\int_1^3 f(x)dx \approx \Delta x\left(f(1.25) + f(1.75) + f(2.25) + f(2.75)\right)$$
$$= \frac{1}{2}\left(1.5 + 1.5 + 2.5 + 3.5\right) = \frac{9}{2}.$$

Problem 4. [20 points] Consider the region trapped by the two curves

$$y = \frac{2}{1+x^2} \text{ and } y = 2 - \sqrt{x}$$

between the points $(0, 2)$ and $(1, 1)$. Here is a sketch showing the region:



Use an integral to express the volume of the solid formed by rotating this region around the y-axis. Do not evaluate the integral.

Answer:
Using shells:

$$V \approx \sum_{i=1}^{n} 2\pi r h \Delta x \Rightarrow V = \int_{x=0}^{x=1} 2\pi x \left(\frac{2}{1+x^2} - (2 - \sqrt{x}) \right) dx.$$

Using washers is a little harder—we need to solve for x in terms of y, which we'll do now:

$$y = \frac{2}{1+x^2} \Rightarrow x = \sqrt{\frac{2}{y} - 1} \text{ and } y = 2 - \sqrt{x} \Rightarrow x = (2-y)^2.$$

Now,

$$V \approx \sum_{i=1}^{n} \pi R^2 - \pi r^2 \Delta y \Rightarrow V = \int_{y=0}^{y=2} \left(\pi \left(\sqrt{\frac{2}{y} - 1} \right)^2 - \pi \left((2-y)^2 \right)^2 . \right) dy$$

$$= \int_{y=1}^{y=2} \pi \left(\frac{2}{y} - 1 - (2-y)^4 \right) dy.$$

It wasn't part of the question, but just for practice, we'll compute these integrals:

$$\int_{x=0}^{x=1} 2\pi x \left(\frac{2}{1+x^2} - (2-\sqrt{x}) \right) dx = 2\pi \left(\ln(1+x^2) + \frac{2}{5} x^{\frac{5}{2}} - x^2 \right) \Big]_0^1 = \left(\ln(4) - \frac{6}{5} \right) \pi \text{ and}$$

$$\int_{y=1}^{y=2} \pi \left(\frac{2}{y} - 1 - (2-y)^4 \right) dy = \pi \left(-17y + 16y^2 - 8y^3 + 2y^4 - \frac{y^5}{5} + 2\ln(y) \right) \Big]_1^2 = \left(\ln(4) - \frac{6}{5} \right) \pi.$$

Problem 5. **[5 points each]** Matching. Put the letter that matches the answer on the line. You need not show your work.

- __(c)__ $\int_{-1}^{3} \frac{dx}{x^2}$

- __(d)__ $\int_{0}^{1} x\sqrt{1-x^2}dx$

- __(a)__ $\int_{-\frac{1}{2}}^{0} 3y\, e^{-2y}\, dy$

- __(b)__ $\int_{-1}^{1} \sqrt{1-t^2}dt$

(a) $-\frac{3}{4}$ (b) $\frac{\pi}{2}$ (c) ∞ (d) $\frac{1}{3}$