Problem 1.

a) Sum of two geometric series. The first terms and the ratios are: $a = 4/25,\ r = 4/5$ and $a = -3/35,\ r = -3/5$, thus both converge and the sum is:

$$s = \frac{4/25}{1-4/5} - \frac{3/35}{1+3/5}$$

b) First

$$\frac{1}{n^2-1} = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1})$$

thus the $N-th$ partial sum is a telescopic sum:

$$S_N = 1/2\,(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{N-2} - \frac{1}{N} + \frac{1}{N-1} - \frac{1}{N+1} = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}$$

The sum of the series is: $S = \lim_{N\to\infty} S_N \ = 1/2 + 1/4 = 3/4$

Problem 2.

a) Using the N-th term Test:

$$\lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Thus the series diverges.

b) Using the Integral Test:

$$\sum_{n=2}^{\infty} \frac{1}{n\,ln(2n)} \geq \int_2^{\infty} \frac{1}{x\,ln(2x)}$$

The improper integral diverges as the antiderivative of $\frac{1}{x\,ln(2x)}$ is $ln\,(ln(2x)) \to \infty$ as $x \to \infty$. Thus the series diverges.

Problem 3.

a) $0 \leq \cos^2(n) \leq 1$, thus: $\cos^2(n)\, n^{-3/2} \leq n^{-3/2}$. The series

$$\sum_{n=1}^{\infty} \cos^2(n) n^{-3/2}$$

converges by the Comparison Test, and the fact that $\sum n^{-3/2}$ converges.

b) Use the Limit Comparison Test, with $b_n = 1/n$:

$$\lim_{n\to\infty} \frac{n+1}{\sqrt{n^4 - n^2 + 1}} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n^2+n}{\sqrt{n^4-n^2+1}} =$$

$$= \lim_{n\to\infty} \frac{1+1/n}{\sqrt{1-1/n^2+1/n^4}} = 1$$

. The series diverges as $\sum 1/n$ diverges.

Problem 4.

a) Use the Ratio Test:

$$\frac{((n+1)^2+1)3^{n+1}}{2^{2n+2}} \Big/ \frac{(n^2+1)3^n}{2^{2n}} = \frac{n^2+2n+2}{n^2+1}\frac{3}{2^2} \to \frac{3}{4} < 1$$

as $n \to \infty$. Thus the series converges.

b) Use the Root Test; the n-th root of the n-th term is

$$\left(\frac{n^{10n}}{2^{n^2}}\right)^{\frac{1}{n}} = \frac{n^{10}}{2^n} \to 0 < 1$$

as exponential functions grow faster than any power. Thus the series converges.

Problem 5.

a) Converges by the Alternating Series Test, as $\frac{1}{n\,ln(n)} \to 0$ and decreasing. As $\sum \frac{1}{n\,ln(n)}$ diverges by the Integral Test, the series converges only Conditionally.

b) Converges by the Alternating Series Test, as $\frac{1}{\sqrt{n^2-n}} \to 0$ and decreasing. As $\sum \frac{1}{\sqrt{n^2-n}}$ diverges by comparing it $\sum \frac{1}{n}$, the series converges only Conditionally.

c) Since $|(-2)^n n^{-2}| = 2^n n^{-2} \to \infty$ as $n \to \infty$, the series diverges by the N-th Term Test.

Problem 6.

a) Using the Ratio Test:

$$\frac{(x-1)^{n+1}}{n+1} \Big/ \frac{(x-1)^n}{n} = \frac{n}{n+1}(x-1) \to x-1$$

, the series converges for $|x-1| < 1$, that is for $0 < x < 2$. The series converges at $x = 0$ (by the alternating series test) and diverges at $x = 2$ (since $\sum 1/n$ diverges).

b) Use the substitution: $u = 4x^2$. Then

$$\frac{1}{1-4x^2} = \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n \, x^{2n}$$