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Problem 1.

The radius of the cross section disc at height y is: $r = \sqrt{100 - y^2}$.

The weight of a thin layer of benzene is: $50\pi(100 - y^2)\, dy$.

The total work needed to pump the liquid out is:

$$W = 50\pi \int_0^{10} (12 - y)(100 - y^2)\, dy = 50\pi \int_0^{10} (y^3 - 12y^2 - 100y + 1200)\, dy = \dots$$

Problem 2.

a) Use integration by parts: $u := \ln(2x)$, $v' := x$, so $u' = 1/(2x)$, $v = x^2/2$, then

$$\int x \ln(2x)\, dx = \frac{x^2 \ln(2x)}{2} - \int \frac{x}{2}\, dx = \frac{x^2 \ln(2x)}{2} - \frac{x^2}{4} + C$$

b) Do first a substitution: $y := x^2$, then $dy = 2x\, dx$, so

$$\int x^3 e^{x^2}\, dx = \frac{1}{2} \int y\, e^y\, dy = \frac{1}{2}\, y\, e^y - \frac{1}{2}\, e^y + C$$

Problem 3.

a) "Split off" a factor: $\sin(2x)$, and substitute: $u = \cos(2x)$. Then one has to express both: $\sin(2x)$ and $\cos(4x)$, in terms of $\cos(2x)$. This is done by:

$$sin^2(2x) = 1 - cos^2(2x) = 1 - u^2$$

$$cos^(2x) = \frac{1 + cos\,(4x)}{2} \quad \Rightarrow \quad cos\,(4x) = 2cos^2(2x) - 1$$

Thus

$$\int sin^3(2x)\, cos\,(4x)\, dx = \frac{-1}{2} \int (1 - u^2)(2u^2 - 1)\, du = \dots$$

b) "Split off" a factor: $\sec^2(3x)$, and substitute: $u = \tan(3x)$. Then the integral takes the form:

$$\frac{1}{3} \int (u^2 + 1) u\, du = \dots$$

Problem 4.

1) (Using trigonometric substitution)

Substitute: $x = 3\,tan\,(\theta)$, then: $\sqrt{4x^2+36} = 6\,sec\,(\theta)$, $dx = 3sec^2(\theta)$, thus one obtains:

$$\int \frac{x}{\sqrt{4x^2+36}}\,dx = \frac{3}{2}\int tan\,(\theta)\,sec\,(\theta)\,d\theta = \frac{3}{2}sec\,(\theta) + C$$

Expressing this in terms of x: $\frac{3}{2}sec\,(\theta) = \frac{1}{2}\sqrt{x^2+9}$.

2) (Using a "simple" substitution)

Let $u = 4x^2 + 36$, then $du = 8x\,dx$ thus

$$\int \frac{x}{\sqrt{4x^2+36}}\,dx = \frac{1}{8}\int u^{-1/2}\,du = \frac{1}{4}u^{1/2} = \frac{1}{4}(4x^2+36)^{1/2} + C$$

Problem 5.

a) First subtracting the denominator from the numerator, one writes:

$$\frac{x^3+1}{x^3-2x^2+x} = 1 + \frac{2x^2-x+1}{x^3-2x^2+x}$$

The denominator factorizes:

$$x^3 - 2x^2 + x = x(x-1)^2$$

thus the form of the partial fraction decomposition is:

$$\frac{2x^2-x+1}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

This gives:

$$2x^2 - x + 1 = A(x-1)^2 + Bx(x-1) + Cx$$

Substitute: $x = 0$, one gets: $A = 1$.

Substitute: $x = 1$, it gives: $C = 2$.

To get B, substitute any other value, p.e.: $x = -1$. This gives: $4 = 4A + 2B - C$ so $B = 1$.

Of course A, B, C can be found also by equating the coefficients of x^2, x and the constant term, and solving the equations obtained.

b) The term $(x+1)^2 + 4$ cannot be factorized. One can compute the integral, by a simple substitution: $u = x + 1$. Then $du = dx$ and the integral becomes:

$$\int \frac{u-1}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du - \int \frac{1}{u^2+4}\,du = \frac{1}{2}\,ln\,(u^2+4) - \frac{1}{2}\,tan^{-1}\left(\frac{u}{2}\right) + C$$