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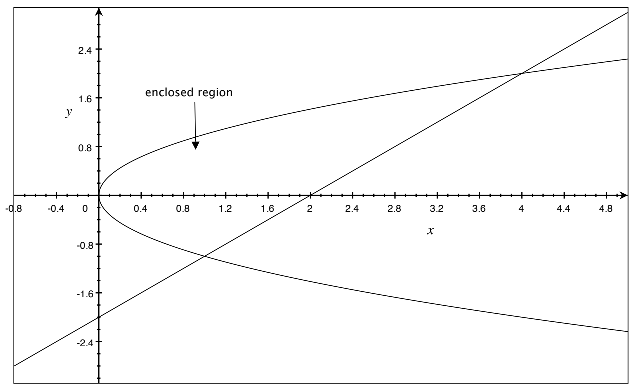
1. (a) (3 pts) Graph the region bounded by $y = x - 2$ and $x = y^2$.

   (b) (12 pts) Find the volume of the solid created by revolving the region bounded by $y = x - 2$ and $x = y^2$ about the line $y = 3$.

**Solution:**

(a) The graph




(b) Here $\Delta V = 2\pi r h$ with $r = 3 - y$ and $h = (y + 2) - y^2$ and so

$$V = \int_{-1}^{2} 2\pi\,(3 - y)\left(y + 2 - y^2\right) dy = 2\pi \int_{-1}^{2} \left(y^3 - 4y^2 + y + 6\right) dy = 2\pi \left[ \frac{y^4}{4} - \frac{4y^3}{3} + \frac{y^2}{2} + 6y \Big|_{-1}^{2} \right] = \frac{45\pi}{2}$$

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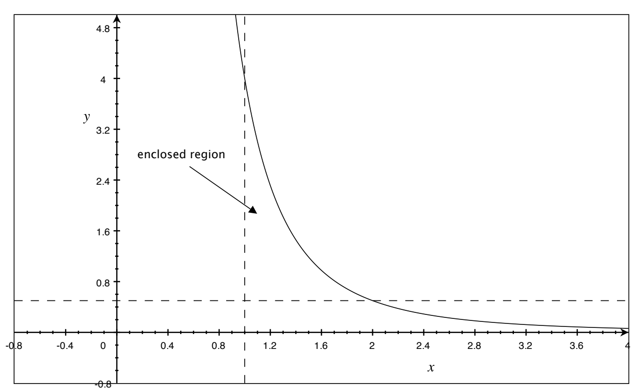
2. (a) (2 pts) Graph the region bounded by the curve $x^3 = 4/y$, and the lines $x = 1$ and $y = 1/2$.

   (b) (8 pts) Set up, but **do not solve**, an integral (or integrals) to find the volume of the solid generated by revolving the region bounded by the curve $x^3 = 4/y$, and the lines $x = 1$ and $y = 1/2$ about the line $x = -e$, using the ***disk/washer method***

**Solution:**

(a) The graph




(b) Here $\Delta V = \pi[R^2 - r^2]$ with $R = \sqrt[3]{4/y} + e$ and $r = 1 + e$ and so

$$V = \int_{1/2}^{4} \pi \left[ \left( \sqrt[3]{4/y} + e \right)^2 - (1 + e)^2 \right] dy$$

3. (10 pts) Solve the following differential equations: (a) $2\sqrt{xy}\dfrac{dy}{dx} = 1$ (b) $\dfrac{dx}{dt} = e^{t-x}$

**Solution:**

(a) Separating yields $y^{1/2}dy = \dfrac{x^{-1/2}}{2}dx$ and so integrating and rearranging terms yields

$$y^{3/2} = \frac{3}{2}x^{1/2} + C, \text{ and so } y = \left(\frac{3}{2}x^{1/2} + C\right)^{2/3}$$

(b) Separating yields $e^x dx = e^t dy$ and so integrating and rearranging terms yields

$$e^x = e^t + C, \text{ and so } x = \ln(e^t + C)$$

4. (20 pts) Consider a thin metal plate with density $\rho(x) = 2$ covering the region bounded by the curve $y = \arctan(x)$, $0 \leq x \leq 1$ and the $x$-axis.

   (a) Find the mass of the thin metal plate.

   (b) Set-up, **but do not solve**, an integral to find the moment about the $x$-axis, $M_x$, of the thin metal plate.

**Solution:**

(a) Note that here $M = \int \rho(x)dA = \int_0^1 2\arctan(x)dx$, where this integral is done using integration by parts with $u = \arctan(x)$ and $dv = dx$, then $du = dx/1 + x^2$ and $v = x$, so

$$M = \int \rho(x)dA = \int_0^1 2\arctan(x)dx \underbrace{=}_{\text{using I.B.P}} 2x\arctan(x)\Big|_0^1 - \int_0^1 \frac{2x}{1+x^2}dx$$

Now, we use a $u$-substitution to solve the last integral with $u = 1 + x^2$, then $du = 2xdx$ and

$$\int \frac{2x}{1+x^2}dx \underbrace{=}_{u=1+x^2} \int \frac{du}{u} = \ln|u| = \ln(1+x^2)$$

so,

$$M = \int_0^1 2\arctan(x)dx = 2x\arctan(x)\Big|_0^1 - \ln(1+x^2)\Big|_0^1 = 2\arctan(1) - \ln(2) = \frac{2\pi}{4} - \ln(2)$$

(b) Here, if we partition the $x$-axis, then the center of each strip is $(\tilde{x}, \tilde{y}) = (x, \arctan(x)/2)$, thus the moment about the $x$-axis is given by

$$M_x = \int \tilde{y}dm = \int \tilde{y}\cdot\rho(x)dA = \int_0^1 \frac{\arctan(x)}{2}\cdot 2\arctan(x)dx = \int_0^1 \arctan^2(x)dx$$

5. (10 pts) Find the length of the curve $y = \cosh(x)$ from $x = 0$ to $x = \ln(2)$.

**Solution:**

Here $ds = \sqrt{1 + y'(x)^2}dx = \sqrt{1 + \sinh^2(x)}dx$ and so

$$L = \int ds = \int_0^{\ln(2)} \sqrt{1 + \sinh^2(x)}dx \underbrace{=}_{\cosh^2(x)=1+\sinh^2(x)} \int_0^{\ln(2)} \sqrt{\cosh^2}dx = \sinh(x)\Big|_0^{\ln(2)} = \sinh(\ln(2)) = \frac{3}{4}$$

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6. (35 pts) Determine if the following converge or diverge. Provide proof of your answers - you must name any test that you use. If any of the series converge, find the sum if possible. (5 pts each)

(a) $\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$ (b) $\left\{\frac{n + \ln(n)}{n}\right\}_{n=1}^{\infty}$ (c) $\left\{\frac{n\sin(n)}{3+n^2}\right\}$ (d) $\sum_{m=2}^{\infty} \frac{e^{2m}}{\pi^m}$

(e) $\{f_m\}_{m=106}^{\infty}$ where $f_m = 1 + (.90)^m$ (f) $\sum_{b=3}^{\infty} \ln(3b) - \ln(2b)$ (g) $\sum_{n=2}^{\infty} \frac{5(-1)^n}{3^{2n}}$

**Solution:** (a) Using partial fractions we have

$$\frac{9}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2}, \text{ so } 3A + 3B = 0, \text{ and } 2A - B = 9 \text{ and so } A = 3, B = -3$$

and so,

$$\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty} \frac{3}{3n-1} - \frac{3}{3n+2} = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \cdots = \frac{3}{2} - \lim_{n\to\infty} \frac{3}{n+2} = \frac{3}{2}$$

thus this is a convergent series which converges to 3/2.

(b) Taking the limit, we have,

$$\lim_{n\to\infty} \frac{n + \ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

so the sequence converges to 1.

(c) Note that $-1 \leq \sin(n) \leq 1$, and

$$\frac{-n}{3+n^2} \leq \frac{n\sin(n)}{3+n^2} \leq \frac{n}{3+n^2}$$

and $\lim_{n\to\infty} \frac{-n}{3+n^2} = \lim_{n\to\infty} \frac{n}{3+n^2} = 0$ (by L'Hopitals rule or dominance of powers), and so by the Squeeze Theorem we have that the sequence converges to 0.

(d) Note that $\frac{e^{2m}}{\pi^m} = \left(\frac{e^2}{\pi}\right)^m$ here we have a Geometric Series with $r = \frac{e^2}{\pi}$ and note that $|r| > 1$ and so this is a divergent series.

(e) Taking the limit we have $\lim_{m\to\infty} f_m = \lim_{m\to\infty} 1 + (0.9)^m \underbrace{=}_{\text{since } 0.9<1} 1$, so the sequence converges to 1.

(f) Note that $\lim_{b\to\infty} a_b = \lim_{b\to\infty} \ln(3b) - \ln(2b) = \lim_{b\to\infty} \ln\left(\frac{3b}{2b}\right) = \ln\left(\frac{3}{2}\right) \neq 0$, and so the series diverges by the Divergence Test

(g) Here we have a geometric with,

$$\sum_{n=2}^{\infty} \frac{5(-1)^n}{3^{2n}} = \sum_{n=2}^{\infty} 5\left(-\frac{1}{3^2}\right)^n = \frac{5}{81} + \frac{5}{81}\left(-\frac{1}{9}\right) + \frac{5}{81}\left(-\frac{1}{9}\right)^2 + \cdots$$

so $a = 5/81$ and $r = -1/9$ and $|r| < 1$ and so

$$\sum_{n=2}^{\infty} \frac{5(-1)^n}{3^{2n}} = \frac{5/81}{1-\left(-\frac{1}{9}\right)} = \frac{5}{90} = \frac{1}{18}$$

thus this is a convergent series which converges to $1/18$.