

1. (30 pts) Determine if the integrals converge or diverge. Each part is worth 10 pts.

(a) $\int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx$ (b) $\int_1^{\infty} \frac{x}{\sqrt{1+x^6}} dx$ (c) $\int_1^{\infty} e^{x^2} dx$

Solution:

(a) Writing the integral as a limit, we have

$$\int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{e^x}{e^{2x} + 1} dx \underset{u=e^x}{=} \lim_{t \rightarrow \infty} \tan^{-1}(e^t) - \tan^{-1}(1) = \pi/2 - \pi/4 = \pi/4$$

so the integral converges to $\pi/4$.

(b) Note that

$$0 < \int_1^{\infty} \frac{x}{\sqrt{1+x^6}} dx < \int_1^{\infty} \frac{x}{\sqrt{x^6}} dx = \int_1^{\infty} \frac{1}{x^2} dx$$

and note that $\int_1^{\infty} \frac{1}{x^2} dx$ converges, and so $\int_1^{\infty} \frac{xdx}{\sqrt{1+x^6}}$ converges by Direct Comparison Test.

(c) Note $x^2 \geq x$ for $x \geq 1$ and so

$$\int_1^{\infty} e^{x^2} \geq \int_1^{\infty} e^x > 0$$

and $\int_1^{\infty} e^x = \lim_{t \rightarrow \infty} e^t - e = \infty$ and so $\int_1^{\infty} e^{x^2}$ diverges by Direct Comparison Test.

2. (16 pts) Find the antiderivative. Each part is worth 8 pts.

(a) $\int \sqrt{1-4x^2} dx$ (b) $\int \frac{\cos(x) \sin(x)}{\sin^2(x) + 3 \sin(x) + 2} dx$

Solution:

(a) Using the substitution $x = \sin(\theta)/2$, we have

$$\begin{aligned} \int \sqrt{1-4x^2} dx &= \frac{1}{2} \int \cos^2(\theta) d\theta = \frac{1}{4} \int [1 + \cos(2\theta)] d\theta \\ &= \frac{1}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right] = \frac{1}{4} \left[\theta + \frac{2 \sin(\theta) \cos(\theta)}{2} \right] = \frac{1}{4} \left[\sin^{-1}(2x) + 2x \sqrt{1-4x^2} \right] + C \end{aligned}$$

(b) If we let $u = \sin(x)$ then $du = \cos(x)dx$ and using partial fractions yields

$$\int \frac{u}{u^2 + 3u + 2} du = \int \frac{2}{u+2} du + \int \frac{-1}{u+1} du = 2 \ln |\sin(x) + 2| - \ln |\sin(x) + 1| + C$$

3. (24 pts) Evaluate the given integrals. Each part is worth 8 pts.

(a) $\int_1^2 xe^x dx$ (b) $\int_{-1}^1 \frac{e^x}{-1+e^x} dx$ (c) $\int_0^{\sqrt{3}} x\sqrt{x^2+1} dx$

Solution:

(a) Using integration by parts with $u = x$ and $dv = e^x dx$ we get,

$$\int_1^2 x e^x dx = x e^x \Big|_1^2 - \int_1^2 e^x dx = e^x(x-1) \Big|_1^2 = e^2$$

(b) Note that the integral is undefined at $x = 0$, so

$$\begin{aligned} \int_{-1}^1 \frac{e^x}{-1+e^x} dx &= \int_{-1}^0 \frac{e^x}{-1+e^x} dx + \int_0^1 \frac{e^x}{-1+e^x} dx \\ &= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{e^x}{-1+e^x} dx + \lim_{t \rightarrow 0^+} \int_t^1 \frac{e^x}{-1+e^x} dx \\ \{u = e^x - 1\} \rightarrow &= \lim_{t \rightarrow 0^-} \ln |e^t - 1| - \ln |e^{-1} - 1| + \lim_{t \rightarrow 0^+} \ln |e^1 - 1| - \ln |e^t - 1| \\ &= \text{"}\infty - \infty\text{"} \end{aligned}$$

so the integral diverges.

(c) Using the u -substitution $u = x^2 + 1$, then $du = 2x dx$ and so

$$\int_0^{\sqrt{3}} x \sqrt{x^2 + 1} dx = \frac{1}{2} \int_1^4 \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^4 = \frac{(4)^{3/2}}{3} - \frac{1}{3} = 7/3.$$

(Alternate method, trigonometric substitution:

Using the substitution $x = \tan(\theta)$, we get

$$\int x \sqrt{x^2 + 1} dx = \int \tan(\theta) \sec(\theta) \cdot \sec^2(\theta) d\theta \underset{u=\sec(\theta)}{=} \int u^2 du = \frac{u^3}{3} = \frac{1}{3} (\sec(\theta))^3 = \frac{(\sqrt{x^2 + 1})^3}{3}$$

so,

$$\int_0^{\sqrt{3}} x \sqrt{x^2 + 1} dx = \frac{(x^2 + 1)^{3/2}}{3} \Big|_0^{\sqrt{3}} = 8/3 - 1/3 = 7/3.$$

4. (a) (9 pts) What is the area of one side of a thin metal plate that covers the region above the x -axis and below the curve $y = \ln(x)$, $e \leq x \leq e^2$.
- (b) (6 pts) Set-up but **do not solve** an integral (or integrals) to find the area of the region bounded by the curves $x^2 + y = 4$, $x + y = 2$ and the lines $x = -2$ and $x = 2$.

Solution:

(a) Here the area is given by $\int_e^{e^2} \ln(x) dx$, now using integration by parts with $u = \ln(x)$ and $dv = dx$ yields

$$\int_e^{e^2} \ln(x) dx = x \ln(x) \Big|_e^{e^2} - \int_e^{e^2} dx = x(\ln(x) - 1) \Big|_e^{e^2} = e^2$$

(b) Here the area is given by

$$A = \int_{-2}^{-1} [(2-x) - (4-x^2)] dx + \int_{-1}^2 [(4-x^2) - (2-x)] dx$$

5. (15 pts) **Always True or False:** Do not justify your answer, just answer either *Always True* or *False*, each part is worth 3 points:

(a) $\int \cos^2(x) dx = \frac{\cos^3(x)}{3} + C$

(b) If $f(x)$ is continuous, then $\int_{-\infty}^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_{-t}^t f(x) dx$

(c) Given the integral $\int \sqrt{x^2 - 1} dx$, one possible trigonometric substitution to use to solve the integral is $x = \tan(\theta)$.

(d) The partial fraction decomposition of $f(x) = \frac{1}{x^2(x^2 + 60)}$ is $\frac{1}{x^2(x^2 + 60)} = \frac{Ax + B}{x^2} + \frac{Cx + D}{x^2 + 60}$.

(e) Suppose, for functions $f(x)$ and $m(x)$, we have that $f(x) \geq m(x) > 0$ for all $x > 106$ then if $\int_{106}^{\infty} m(x)$ diverges then so does $\int_{106}^{\infty} f(x)$.

Solution:

(a) False (b) False (c) False (d) True (e) True
