

1. Evaluate  $\int_2^3 x e^{x^2} dx$ .

Call the integral  $I$ , and let  $t = x^2$ . Then  $dt = 2x dx$ .

When  $x = 3$ ,  $t = 9$  and when  $x = 2$ ,  $t = 4$ . Therefore

$$I = \frac{1}{2} \int_4^9 e^t dt = \frac{1}{2} e^t \Big|_4^9 = \frac{e^9}{2} - \frac{e^4}{2}.$$

2. Evaluate  $\int \frac{9x + 9}{(x - 1)(x^2 + 4x + 13)} dx$ .

The second factor is irreducible. Set up partial fractions:

$$\frac{9x + 9}{(x - 1)(x^2 + 4x + 13)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4x + 13},$$

so

$$9x + 9 = A(x^2 + 4x + 13) + (Bx + C)(x - 1).$$

Choosing  $x = 1$  we find  $18 = A(18)$ , so  $A = 1$ . Hence

$$9x + 9 = x^2 + 4x + 13 + (Bx + C)(x - 1) = (1 + B)x^2 + (4 - B + C)x + (13 - C).$$

Equating coefficients, we find

$$\begin{aligned} 0 &= 1 + B \\ 9 &= 4 - B + C \\ 9 &= 13 - C \end{aligned}$$

Thus  $B = -1$  and  $C = 4$ . Call the integral  $I$ . Then

$$I = \int \frac{1}{x - 1} dx + \int \frac{-x + 4}{x^2 + 4x + 13} dx.$$

The first integral is  $\ln|x - 1|$  plus a constant. Call the second integral  $J$ .

To find  $J$ , complete the square:  $x^2 + 4x + 13 = (x + 2)^2 + 9$ . Now let  $t = x + 2$ , so  $dx = dt$  and  $x = t - 2$ . Hence

$$J = \int \frac{-(t - 2) + 4}{t^2 + 9} dt = - \int \frac{t}{t^2 + 9} dt + \int \frac{6}{t^2 + 9} dt = -\frac{1}{2} \ln(t^2 + 9) + \frac{6}{3} \tan^{-1} \frac{t}{3} + C.$$

The final answer is

$$\ln|x - 1| - \frac{1}{2} \ln((x + 2)^2 + 9) + 2 \tan^{-1} \frac{x + 2}{3} + C.$$

3. Evaluate  $\int e^x \sin x \, dx$ .

Call the integral  $I$  and integrate by parts. Choose  $u = e^x$ . Then

$$\begin{aligned} u &= e^x & dv &= \sin x \, dx \\ du &= e^x \, dx & v &= -\cos x \end{aligned}$$

Therefore

$$I = -e^x \cos x + \int e^x \cos x \, dx.$$

Integrate by parts again. Choose  $U = e^x$ . Then

$$\begin{aligned} U &= e^x & dV &= \cos x \, dx \\ dU &= e^x \, dx & V &= \sin x \end{aligned}$$

Then

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - I.$$

Solving for  $I$  by algebra, we find

$$I = -\frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + C.$$

4. Evaluate  $\int \frac{\cos x \, dx}{(\sin^2 x + 4)^{5/2}}$ .

Call the integral  $I$  and let  $t = \sin x$ , so  $dt = \cos x \, dx$ . Then

$$I = \int \frac{dt}{(t^2 + 4)^{5/2}}.$$

Now use trigonometric substitution: let  $t = 2 \tan \theta$ , so  $dt = 2 \sec^2 \theta$  and  $(t^2 + 4)^{5/2} = 32 \sec^5 \theta$ . Hence

$$\begin{aligned} I &= \frac{1}{16} \int \frac{\sec^2 \theta}{\sec^5 \theta} d\theta = \frac{1}{16} \int \cos^3 \theta d\theta = \frac{1}{16} \int (1 - \sin^2 \theta) \cos \theta d\theta \\ &= \frac{1}{16} \int \cos \theta d\theta - \frac{1}{16} \int \sin^2 \theta \cos \theta d\theta = \frac{\sin \theta}{16} - \frac{\sin^3 \theta}{48} + C. \end{aligned}$$

Draw a right triangle with angle  $\theta$ , label the opposite side  $t$  and the adjacent side 2, so  $\tan \theta = \frac{t}{2}$ . Then the hypotenuse is  $\sqrt{t^2 + 4}$ , so  $\sin \theta = \frac{t}{\sqrt{t^2 + 4}}$ , and

$$I = \frac{1}{16} \frac{t}{\sqrt{t^2 + 4}} - \frac{1}{48} \left( \frac{t}{\sqrt{t^2 + 4}} \right)^3 + C = \frac{1}{16} \frac{\sin x}{\sqrt{\sin^2 x + 4}} - \frac{1}{48} \left( \frac{\sin x}{\sqrt{\sin^2 x + 4}} \right)^3 + C.$$

5. Does the following integral converge or diverge? Give your reasons fully and clearly. If the integral converges, find its value.

$$\int_{\pi/4}^{\pi/2} \frac{\sec^2 \theta}{\tan^2 \theta - 1} d\theta$$

Call the integral  $I$  and let  $t = \tan \theta$ . Then  $dt = \sec^2 \theta d\theta$ . When  $\theta \rightarrow \pi/2$ ,  $t \rightarrow \infty$  and when  $\theta = \pi/4$ ,  $t = 1$ . Therefore

$$I = \int_1^{\infty} \frac{dt}{t^2 - 1} dt.$$

There are two bad points: 1 and  $\infty$ , so write  $I = I_1 + I_2$  where

$$I_1 = \int_1^2 \frac{dt}{t^2 - 1}$$
$$I_2 = \int_2^{\infty} \frac{dt}{t^2 - 1}$$

Now  $t^2 - 1 = (t + 1)(t - 1)$  so  $\frac{1}{t^2 - 1} \sim \frac{1}{2(t - 1)}$  as  $t \rightarrow 1$  (since  $t + 1 \rightarrow 2$  as  $t \rightarrow 1$ ). Let  $y = t - 1$ , so  $dt = dy$ . When  $t = 2$ ,  $y = 1$  and when  $t = 1$ ,  $y = 0$ . Thus

$$I_1 = \frac{1}{2} \int_0^1 \frac{dy}{y},$$

which diverges by the  $p$ -test at 0 ( $p = 1 \geq 1$ ). Therefore  $I_1$  diverges, and so  $I$  diverges (even though  $I_2$  converges).

6. Does the following series converge or diverge? Give your reasons fully and clearly. If the series converges, find its value.

$$\sum_{n=1}^{\infty} e^{-2\pi n}$$

This is a geometric series with ratio  $e^{-2\pi}$ , and  $|e^{-2\pi}| < 1$ , so the series converges. It begins with  $n = 1$  rather than  $n = 0$ , so its value is

$$\frac{e^{-2\pi}}{1 - e^{-2\pi}}.$$

7. Does the following integral converge or diverge? Give your reasons fully and clearly.

$$\int_1^{\infty} \frac{dx}{e^{-x} + \sqrt{x-1}}$$

Call the integral  $I$ . The only bad point is  $\infty$  and

$$\frac{1}{e^{-x} + \sqrt{x-1}} \sim \frac{1}{\sqrt{x}}$$

as  $x \rightarrow \infty$ . But

$$\int_1^{\infty} \frac{dx}{\sqrt{x}}$$

diverges by the  $p$ -test at  $\infty$  ( $p = 1/2 \leq 1$ ), so  $I$  diverges by limit comparison.

8. Does the following series converge or diverge? Give your reasons fully and clearly.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Use the integral test. Let  $I = \int_2^{\infty} \frac{dx}{x \ln x}$ . The integrand is continuous and decreasing for  $2 \leq x < \infty$ , so the series and the integral both converge or both diverge.

Let  $t = \ln x$ , so  $dt = \frac{dx}{x}$ . When  $x \rightarrow \infty$ ,  $t \rightarrow \infty$  and when  $x = 2$ ,  $t = \ln 2$ . Hence

$$I = \int_{\ln 2}^{\infty} \frac{dx}{x},$$

which diverges by the  $p$ -test at  $\infty$  ( $p = 1 \leq 1$ ). Therefore the series diverges by the integral test.