1. $\int_0^1 \frac{\sin(\tan^{-1} x) \tan^{-1} x}{x^2+1}\ dx = \frac{1}{\sqrt{2}}(1 - \frac{\pi}{4})$

$u = \tan^{-1} x$

$du = \frac{dx}{x^2+1}$

$$\begin{aligned}\int_0^1 \frac{\sin(\tan^{-1} x) \tan^{-1} x}{x^2+1}\ dx &= \int_0^{\pi/4} u \sin u\ du \\ &= -u\cos u + \sin u \Big|_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}}(1 - \frac{\pi}{4})\end{aligned}$$

2. $\int \frac{x^2 - x + 2}{x^3 + x}\ dx = 2\ln|x| - 1/2\ln(x^2+1) - \tan^{-1} x + C$

$$\frac{x^2 - x + 2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2+1}$$
$$x^2 - x + 2 = A(x^2+1) + (Bx + C)x$$
$$A = 2 \quad C = -1 \quad B = -1$$

$$\begin{aligned}\int \frac{x^2 - x + 2}{x^3 + x}\ dx &= \int \frac{2}{x}\ dx - \int \frac{x}{x^2+1}\ dx - \int \frac{dx}{x^2+1} \\ &= 2\ln|x| - 1/2\ln(x^2+1) - \tan^{-1} x + C\end{aligned}$$

3. $\int e^{\frac{\sqrt{1-x^2}}{x}} \cdot \frac{dx}{x^2\sqrt{1-x^2}} = -e^{\frac{\sqrt{1-x^2}}{x}} + C$

$x = \sin\theta$

$dx = \cos\theta\ d\theta.$

$$\begin{aligned}\int e^{\frac{\sqrt{1-x^2}}{x}} \cdot \frac{dx}{x^2\sqrt{1-x^2}} &= \int e^{\frac{\cos\theta}{\sin\theta}} \cdot \frac{\cos\theta\ d\theta}{\sin^2\theta\cos\theta} \\ &= \int e^{\cot\theta} \cdot \csc^2\theta\ d\theta \\ &= -e^{\cot\theta} + C \\ &= -e^{\frac{\sqrt{1-x^2}}{x}} + C\end{aligned}$$

4. $\int_0^{\pi} \frac{\sin\sqrt{x}}{x(1+x)}dx$ Converges.

As $x \to 0$, $\sin x \sim x$, so $\sin\sqrt{x} \sim \sqrt{x}$.

$$\frac{\sin\sqrt{x}}{x(1+x)} \sim \frac{\sqrt{x}}{x \cdot 2} = \frac{1}{2x^{1/2}}.$$

$\int_0^{\pi} \frac{dx}{2x^{1/2}}$ converges, so by limit comparison test, the original integral converges.

5. $\int_0^{\infty} \frac{\ln(5+\cos x)}{x^2}dx$ Diverges.

$$\int_0^{\infty} \frac{\ln(5+\cos x)}{x^2} = \int_0^1 \frac{\ln(5+\cos x)}{x^2} + \int_1^{\infty} \frac{\ln(5+\cos x)}{x^2}$$

As $x \to 0$, $\frac{\ln(5+\cos x)}{x^2} \sim \frac{\ln 6}{x^2}$. $\int_0^1 \frac{\ln 6}{x^2}\,dx$ diverges, so first integral diverges by limit comparison test. Since one part of the original integral diverges, the whole integral must diverge.

6. $\sum_{n=3}^{\infty} \frac{7^n \ln n}{n!}$ Converges.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{7^{n+1}\ln(n+1)}{(n+1)!} \cdot \frac{n!}{7^n \ln n} \\
&= \lim_{n\to\infty} \frac{7}{n+1} \cdot \frac{\ln(n+1)}{\ln n} \\
&= 0 < 1
\end{aligned}$$

so series converges by Ratio Test.

7. $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2}) \ln n$ Converges.

As $n \to \infty$, $1/n^2 \to 0$, so $\sin(1/n^2) \sim 1/n^2$. $\ln n < n^1/2$, so

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} < \sum_{n=1}^{\infty} \frac{n^{1/2}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

The right hand series converges, so the left hand series does as well. (Direct Comparison.) Then the original series converges, by Limit Comparison.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n + 3^{n+1}}{5^n} = \frac{89}{30}$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{2^n + 3^{n+1}}{5^n} &= \sum_{n=2}^{\infty} \frac{2^n}{5^n} + \sum_{n=2}^{\infty} \frac{3^{n+1}}{5^n} \\
&= (\frac{4}{25} + \frac{8}{125} + \ldots) + (\frac{27}{25} + \frac{81}{125} + \ldots) \\
&= \frac{4}{25}(1 + \frac{2}{5} + \frac{4}{25} + \ldots) + \frac{27}{25}(1 + \frac{3}{5} + \frac{9}{25} + \ldots) \\
&= \frac{4}{25} \cdot \frac{1}{1 - 2/5} + \frac{27}{25} \cdot \frac{1}{1 - 3/5} \\
&= \frac{89}{30}
\end{aligned}$$

9. $\displaystyle\sum_{n=1}^{\infty} (1 + \frac{x}{n})^{n^2}$ converges when $x < 0$.

Use the root test.

$$\begin{aligned}
\lim_{n\to\infty} a_n^{1/n} &= \lim_{n\to\infty} (1 + x/n)^{n^2/n} \\
&= \lim_{n\to\infty} (1 + x/n)^n \\
&= e^x.
\end{aligned}$$

When this limit is < 1, the series converges. Thus the series converges when $e^x < 1$, *i.e.* when $x < 0$. When this limit is > 1, the series diverges. Thus the series diverges when $e^x > 1$, *i.e.* when $x > 0$. When the limit is equal to 1 *i.e.* when $x = 0$, the test doesn't tell us anything, but in this case, the series is

$$\sum_{n=1}^{\infty} (1)^{n^2} = \sum_{n=1}^{\infty} 1$$

which obviously diverges.