Section A. Work all problems

1. Differentiate each of the following:

a. $f(x) = 5x^3 + \sqrt{x}$

(*) $15x^2 + \frac{1}{2\sqrt{x}}$

b. $f(x) = (x^5 + 1)^{10}$

(*) $50x^4 \left(x^5 + 1\right)^9$

c. $f(x) = x^3 e^x$

(*) $e^x x^3 + 3e^x x^2$

d. $f(x) = \frac{x^2+1}{2x-6}$

(*) $\frac{2x}{2x-6} - \frac{2\left(x^2+1\right)}{(2x-6)^2}$

e. $f(x) = e^{x^2+1}$

(*) $2xe^{x^2+1}$

f. $f(x) = \ln(x^2 + x + 5)$

(*) $\frac{2x+1}{x^2+x+5}$

g. $f(x) = e^{3x+1}$

(*) $3e^{3x+1}$

h. $f(x) = \ln(3x)$

(*) $\frac{1}{x}$

i. $f(x) = 3x^2 - x - 2$

(*) $6x - 1$

j. $f(x) = (x^3 + x)^9$

(*) $9\left(3x^2 + 1\right)\left(x^3 + x\right)^8$

k. $f(x) = e^x(x^2 + 1)$

(*) $2e^x x + e^x\left(x^2 + 1\right)$

l. $f(x) = e^x + 2$

(*) e^x

m. $f(x) = \ln(x^2 + 1)$

(*) $\frac{2x}{x^2+1}$

n. $f(x) = 2000e^{2x}$

(*) $4000e^{2x}$

o. $f(x) = x^2 + \sqrt{x}$

(*) $2x + \frac{1}{2\sqrt{x}}$

p. $f(x) = \frac{x^2+2}{x-9}$

(*) $\frac{x^2-18x-2}{(x-9)^2}$

q. $f(x) = 6x + 2$ *Use the limit definition of the derivative*

(*) Difference quotient is 6.
Take the limit as h approaches 0 to get 6.

2. Compute each of the following:

a. $\int 6x^3 + 6\sqrt{x}dx$

(*) $\frac{3x^4}{2} + 4x^{3/2} + C$

b. $\int \frac{5}{x}dx$

(*) $5\ln|x| + C$

c. $\int e^x + 2e^x dx$

(*) $3e^x + C$

d. $\int \frac{6x^2+5}{2x^3+5x+1} dx$

(*) $\log\left(2x^3+5x+1\right)+C$

e. $\int 6xe^{x^2+1} dx$

(*) $3e^{x^2+1}+C$

f. $\int 15x^2(x^3+2)^9 dx$

(*) $.5(x^3+2)^{10}+C.$

g. $\int_0^1 x^2+x dx$

(*) $\frac{5}{6}$

h. $\int_1^2 \frac{x-1}{x^2-x} dx$

(*) $-$

i. $\int x^2-x-1 dx$

(*) $\frac{1}{6}x\left(2x^2-3x-6\right)+C$

j. $\int \frac{90}{x} dx$

(*) $90\log(x)+C$

k. $\int 30x^2(10x^3+55)^6 dx$

(*) $\frac{1}{7}(10x^3+55)^7+C$

l. $\int \frac{4x+14}{x^2+7x+100} dx$

(*) $2\log\left(x^2+7x+100\right)+C$

m. $\int_1^2 \sqrt{x} dx$

(*) 1.21895

n. $\int_1^2 e^{6x} dx$

(*) 27058.6

o. $\int_1^2 -2xe^{-x^2} dx$

(*) -0.349564

3. Consider the difference equation $y_{n+1}=-\frac{3}{2}y_n+5, y_0=-11$.

a. Is the difference equation constant? How do you know?

b. Is the difference equation oscillating or monotonic? How do you know?

c. Is the difference equation bounded or unbounded? How do you know?

d. Compute the first two terms in the sequence, y_1 and y_2.

e. Write down the general solution of the difference equation, $y_n =$.

f. Compute y_{99}.

4. Consider the function $f(x) = x^2 + x + 1$.

a. Compute the *difference quotient* of $f(x)$

> **(*)** The difference quotient is $\frac{f(x+h)-f(x)}{h}$. You should obtain $2x + h + 1$.
>
> Note that if you let h approach zero, you get $2x + 1$, which is the derivative.

b. Compute the *instantaneous rate of change* of $f(x)$ at $x = 2$.

> **(*)** $f'(x) = 2x + 1$.
> $f'(2) = 5$.

c. Consider the line tangent to $f(x)$ at $x = 2$. Write the equation of a line parallel to the tangent line, with y intercept 2.

> **(*)** Recall the slope of the tangent line at x is $f'(x)$. I want a line with slope 5 (same slope as that of the tangent line), and y intercept 2. This is $y = 5x + 2$.

5. Evaluate each of the following limits. If the limit does not evaluate to a finite real number, write "DNE."

a. $\lim_{x\to 4} \frac{x^2-2x-8}{x+2}$

b. $\lim_{x\to 0} \sqrt{x^2+x+1}$

c. $\lim_{x\to\infty} \frac{-x+6}{x^2+3}$

6. Let $f(7) = 1$, $f'(7) = 2$, $g(7) = 5$, and $g'(7) = 9$.

a. Find $h(7)$ and $h'(7)$ where $h(x) = f(x) + 3g(x)$.

b. Find $h(7)$ and $h'(7)$ where $h(x) = \frac{1}{g(x)}$.

c. Find $h(7)$ and $h'(7)$ where $h(x) = \frac{f(x)}{g(x)}$.

d. Find $h(7)$ and $h'(7)$ where $h(x) = f(x)g(x)$.

e. Find $h(7)$ and $h'(7)$ where $h(x) = f(g(x))$.

f. Find $h(7)$ and $h'(7)$ where $h(x) = \frac{x}{g(x)}$.

Section B. On the actual Final you will be able to drop one question

1. Ira wishes to set up a retirement account. At age 25 she makes an initial deposit of $1,000. The account's annual interest rate is 9%, compounded monthly. She wants to determine what her monthly deposit should be in order to obtain $1,000,000 by the time she is age 65.

   a. Write down the explicit difference equation $y_{n+1} = ay_n + b$ pertaining to this scenario.

   (*) Note that annual interest is 9%, so monthly interest is .75%, or .0075. Then $a = 1 + .0075 = 1.0075$.

   $y_{n+1} = (1.0075)y_n + b$

   b. Write down the general solution of the difference equation, $y_n =$.

   (*) $y_n = \frac{b}{-.0075} + (1.0075)^n(1000 - \frac{b}{-.0075})$.

   c. Determine her monthly deposit so that she reaches her goal in time.

   (*) In 40 years, 480 compounding terms will have passed. She wants 1,000,000 dollars then. So let $n = 480$ and $y_{480} = 1000000$. Plug these values into the general solution (part b), and solve for b.

   $b = 205.901$.

   d. Now suppose Ira is 65 and has reached her goal. She has stopped her monthly contributions. Consider the same interest rate and compounding. What is the maximum she can *withdraw* per month so that her savings last indefinitely?

   (*) A difference equation is constant if $y_0 = \frac{b}{1-a}$. We know the value for a. When Ira is 65, $y_0 = 1000000$. Solve for b to get $b = -7500$. This means she can withdraw 7500 per month.

2. Let $f(x) = \frac{2}{3}x^3 - 18x$

   a. Locate the relative extrema on $f(x)$.

   (*) $f'(x) = 2x^2 - 18$.

   Solve $f'(x) = 0$ to get $x = 3, x = -3$.

   b. Classify the relative extrema as relative maxima, minima, or neither.

   c. Identify the interval(s) for which $f(x)$ is increasing.

   d. Locate the inflection point(s) on $f(x)$.

   e. Identify the interval(s) for which $f(x)$ is concave down.

   f. Find the absolute max and min of $f(x)$ in the domain $[-4, 2]$

3. A developer is considering buying a rectangular section of land alongside a lake for the site of a condominium. The city determines the property tax in the following way. Each foot of land bordering the lake incurs a tax of 30 dollars per month. Each foot of land alongside the other edges incurs a tax of 15 dollars per month. The developer wants to determine the largest area of land she can buy so that the monthly property tax does not exceed 5000 dollars. Determine the optimal dimensions of the site.

4. The revenue earned by producing and selling x units of the board game *Watsonopoly* can be modelled by the revenue function

$$R(x) = 1700 - .09x - 1000(.2x + 7)^{-1}$$

a. Compute a function $R'(x)$ which gives the *marginal revenue* as a function of x.

> **(*)** $R'(x) = \frac{200}{(.2x+7)^2} - .09$.

b. Suppose I am producing 300 units. Should I expect revenues to increase or decrease by increasing production? Justify your answer.

> **(*)** $R'(300) < 0$, so I should expect revenue to decrease by increasing production.

5. Compute $\int_1^\infty \frac{1}{x^2}$.

6. Compute $\int_1^\infty \frac{1}{x^x}$.

7. In October 2009 Milton's consumption of cat litter is 40lb per month, and is increasing at the rate of 2lb per month. The cost per pound is .75 dollars, and is increasing at the rate of .1 dollars per month. What is Milton's total cat litter bill for the month? At what rate is it increasing?

8. The bacteria known to cause deadly Watsonitis are being studied. A scientist places $1,000$ bacteria in a dish. Six hours later, the number of bacteria has increased to $9,300$. It is known that the population of bacteria grow according to the model $\frac{dP}{dt} = kP(t)$, where $P(t)$ is the population after t hours. Construct the function $P(t)$ which gives the population after t hours and satisfies the differential equation given. Use your function to estimate the population of the colony after one day (24 hours).

9. Archaeologists believe they have found the lost continent of Atlantis. To help test their conjecture, they obtain samples from bones found at the site. They determine the bones to have 42% of their original amount of Carbon-14. Assuming a half-life of $5,700$ years and that Carbon-14 decays exponentially, determine the age of the bones.

10. **a.** Solve the variables-separable differential equation $\frac{dy}{dt} = -ky$.

> **(*)** Recall we had the theorem $\frac{dy}{dt} = -ky$ if and only if $y = Ce^{-kt}$.

b. Watsonium-X decays at a rate of 4% per year. Find its half life.

> **(*)** The half-life is the time required for $\frac{1}{2}$ of the current amont to decay. In this problem we're given $k = .04$. (Note that usually we are given the half life, and have to find the decay constant. In this problem we have the opposite scenario). Recall C is the initial amount. By the definition of half-life, I'm looking for the value of t so that $y = .5C$.
>
> Solve $.5C = Ce^{-.04t}$. You should get $t \approx 17.3287$.

c. Archaeologists who have found *The Milton Scrolls* have determined they contain 25% of their starting amount of Watsonium-X. How old are the scrolls?

> **(*)** Not very old. Watsonium-X decays quickly, as we've seen above. I want to find the value of t so that $.25C = Ce^{-.04t}$.
>
> You should get $t \approx 34.6574$.
>
> Note that this is twice the half life. If it takes about 17 years for half to decay, then it'll take another 17 years for half of what remains to decay. So after about 34 years we're left with a quarter of the original amount.

11. Let $f(x) = e^3 + \sqrt{x}$.

    a. Approximate the area under the curve $f(x)$, bounded on the left by $x = 0$, and on the right by $x = 1$, using Riemann Sums. Use the left-handed rule and 2 rectangles.

    b. Find the exact area of the region described.

12. Let $f(x) = 4xe^{x^2}$

    a. Compute $F(x)$, the anti-derivative of $f(x)$.

    b. Let a region be bounded by the x-axis, $f(x)$, $x = 1$, and $x = 2$. Compute the volume of the solid obtained by rotating this region about the x-axis.

13. Compute the area bounded by the curve $f(x) = 1 - x$, the x-axis, $x = 0$ and $x = 2$.

14. Find the area of the region bounded by the curves $f(x) = \sqrt{x}$ and $g(x) = x$.

15. Consider $f(x) = \frac{1}{x}$

    a. Find the area of the region bounded by $f(x)$, the x-axis, $x = 1$ and $x = 2$.

    b. Find the volume obtained by rotation the above region about the x-axis.

    c. Compute $\int_1^\infty f(x)dx$.

16. Watsonium-Q is a fuel used for rockets in the late 21st century. Usage is given in tons according to the formula $A(t) = 33000e^{.04t}$, where t is years since 2050.

    a. What does the area under the curve represent with respect to this problem?

    b. Calculate the total usage of Watsonium-Q over the years 2050 to 2100.

    c. Suppose worldwide reserves of Watsonium-Q is a fixed 2.1 million tons. Assuming no future growth in reserves, is 2.1 million tons enough to meet the predicted demand over the years 2050 - 2100?

    d. Suppose in 2100 the rate of increase in usage declined from 4% to 3%. Write a new model for $A(t)$ expressing this change.

    e. Calculate the amount of Watsonium-Q saved from 2100 to 2120 by computing the difference in the total usage predicted by the two formulas.

17. Watsonovia Bank offers an account which pays interest at the rate of 6%, compounded continuously.

    a. I deposit $10,000$ dollars into an account. What is its value after 20 years?

    b. Find the future value of a continous money flow of $1,000$ dollars per year for 40 years.

    c. Find the future value of an investment over a 30 year period if there is a continuous money flow of $3,000$ dollars per year.

18. The birth rate $b(t)$ and death rate $d(t)$ for Watson City is given as the following functions of t (years since 2000).

$$b(t) = 733 - 8t^2, \text{ and } d(t) = 247 + 17t^2$$

a. Find the total number of births from the years 2000 to 2015.

b. Find the total number of deaths from the years 2000 to 2015.

c. The population in the year 2000 was 181,000. Estimate the population in the year 2015.

19. Let $f(x) = x^2 + x$. Suppose I want to approximate $\int_1^5 f(x)dx$ using Riemann Sums and the Midpoint rule. It is known that the error is at most

$$\frac{A(b-a)^3}{12n^2}$$

a. Compute $f''(x)$.

b. What is the lowest possible value for A so that $|f''(x)| \leq A$?

c. What should be the number of partitions n so that my error is at most $\frac{1}{1000}$