1. Definitions and Concepts.

a. Give a function, f, which satisfies the relationship $\frac{df}{dx} = 1$.

(*) Another way to interpret the relationship is $f'(x) = 1$, then $f(x) = x + C$, the anti-derivative.

b. Why does it not matter if I forget the "+C" when using *definite* integration?

(*) Here is an example of what will happen. Let's integrate $\int_0^2 f(x)dx$ with $f(x) = 3x^2$.

The definite integral is computed as $F(b) - F(a)$ where $F(x)$ is the anti-derivative of $f(x)$. The anti-derivative of $3x^2$ is $F(x) = x^3 + C$.

Now $F(2) - F(0) = (2^3 + C) - (0^3 + C) = 8 + C - C = 8$. See how the C's canceled out? That should always happen. So it is the convention to just drop the '+C' altogether. Sometimes this can get you into trouble! (But not in anything we're doing in this class).

c. Find the error in my proof that $0 = 1$.

Let $f(x) = 2x$ and $g(x) = 2x+1$. Then $f'(x) = 2$ and $g'(x) = 2$. Anti-differentiating, $f(x) = 2x+C$ and $g(x) = 2x + C$. Then $f(x) = g(x)$. So it follows $2x = 2x + 1$. Subtrating $2x$ from both sides of the equation, I find $0 = 1$.

(*) C is not a variable. We don't know that the values of C in $f(x)$ and $g(x)$ are the same.

e. Give an example of a function, $f(x)$ such that the area bounded by the x-axis, $f(x)$, $x = 0$, and $x = 1$ can be determined exactly using one rectangle (as opposed to an infinite series of rectangles).

(*) Let $f(x) = 3$. Then the region itself is a rectangle!

f. Give an example of a function, $f(x)$, such that $\int f(x) = 3x^2 + 2x$.

(*) I want a function such that its anti-derivative is $3x^2 + 2x$. So I take the derivative of $3x^2 + 2x$. This is $6x + 2$.

2. Find $\int f(x)dx$ for each of the following.

a. $f(x) = 3x^2 + 5x + 1$

(*) Using the power rule for anti-derivatives (add one to power, then divide by new power), I get $F(x) = x^3 + \frac{5}{2}x^2 + x + C$.

b. $f(x) = \frac{44}{x}$

(*) $\int \frac{44}{x}dx = 44\int \frac{1}{x}dx = 44\ln|x| + C$

c. $f(x) = \frac{1}{x}(\ln(3x))^5$

(*) Here is a substitution problem.

Let $u = \ln(3x)$, the "inside" function. Then $\frac{du}{dx} = \frac{1}{x}$. So we see the following:

$$\int \frac{1}{x}(\ln(3x))^5 dx = \int \frac{du}{dx}u^5 dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}(\ln(3x))^6 + C$$

d. $f(x) = \frac{9x^2+3}{3x^3+3x}$

(*) Let $u = 3x^3 + 3x$. Then $\frac{du}{dx} = 9x^2 + 3$. So we see the following:

$$\int \frac{9x^2+3}{3x^3+3x}dx = \int \frac{du}{dx}\frac{1}{u}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|(3x^3 + 3x)| + C$$

e. $f(x) = \sqrt{x}$

(*) Notice $f(x) = x^{\frac{1}{2}}$. Then, using the power rule for anti-derivatives, $F(x) = \frac{2}{3}x^{\frac{3}{2}} + C$.

3. For each of the following, use Riemann Sums (with "left-hand rule") with 4 rectangles to approximate the area under $f(x)$, bounded on the left by 0 and on the right by 2. Then find $\int_0^2 f(x)dx$ for each of the following. Finally, give the average value of the function over the interval from 0 to 2.

a. $f(x) = x^2 + x$

(*) With $n = 4$, I have 4 rectangles in an interval of width 2. This means each rectangle has width .5 (2 divided by 4). To use the left-hand rule, I consider the x-coordinate of the left side of each rectangle. The first rectangle has left x-coordinate 0. The second rectangle has left x-coordinate .5. And so on. If I let x_i be the left x-coordinate of each rectangle, I have

$x_1 = 0, x_2 = .5, x_3 = 1, x_4 = 1.5.$

Now to get the height of each rectangle I plug in the x-coordinate into the function. I have

$f(x_1) = f(0) = 0, f(x_2) = f(.5) = .75, f(x_3) = f(1) = 2, f(x_4) = f(1.5) = 3.75.$

The area of a rectangle is width $\times$ height. So the area of rectangle 1 is $0 \times .5$, or 0. The areas of the four rectangles are given:

$A_1 = 0, A_2 = .375, A_3 = 1, A_4 = 1.875.$

To get the total area I add up the areas of all four rectangles. This is 3.25.

To find the definite integral, note I have $f(x) = x^2 + x$. Then $F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2$. Then $F(2) - F(0) = \frac{8}{3} + 2 = \frac{14}{3}$.

Finally, the average value of a function is given by $\frac{1}{b-a}\int_a^b f(x)dx$. So the average value of $f(x)$ above is $\frac{1}{2} \times \frac{14}{3} = \frac{14}{6}$.

b. $f(x) = xe^{x^2}$

(*) I have the same interval with the same number of rectangles, so the x-coordinates of the rectangles are the same as above: $x_1 = 0, x_2 = .5, x_3 = 1, x_4 = 1.5$.
I get the areas of the rectangles by multiplying $f(x_i)$ by .5 for $i = 1, 2, 3, 4$.
Area of Rectangle 1 = $.5 \times 0e^0 = 0$.
Area of Rectangle 2 = $.5 \times .5e^{.25}$.
Area of Rectangle 3 = $.5 \times 1e^1$.
Area of Rectangle 4 = $.5 \times 1.5e^{2.25}$.

To get the total area, sum up the areas of the four rectangles.

To find the definite integral, we consider $f(x) = xe^{x^2}$. Finding $F(x)$ will require u-substitution. First let $u = x^2$. Then $\frac{du}{dx} = 2x$. Solve for dx to see $dx = \frac{du}{2x}$.

Originally $F(x) = \int xe^{x^2}dx$. But substituting u for x^2 and $\frac{du}{2x}$ for dx, we get $\int xe^u \frac{du}{2x}$.

The x's cancel so this simplifies to $\int \frac{1}{2}e^u du$, which evaluates to $\frac{1}{2}e^u$ since the anti-derivative of e^u is e^u. Finally, now that we're done with the integration, let $u = x^2$ again. So $F(x) = \frac{1}{2}e^{x^2}$.

$F(2) - F(0) = \frac{1}{2}e^4 - \frac{1}{2}$.

Recall the average value formula explained above. The average value is $\frac{1}{2}(\frac{1}{2}e^4 - \frac{1}{2})$.

4. Find the area of the region bounded by the x-axis and the curve $f(x) = -(x^2) + 4$.

(*) First I notice that the limits of integration have not been given to me, but I know that the region will be bounded at the points where the curve crosses the x-axis. So first I solve $f(x) = 0$. This gives me $x = \pm 2$.

If $f(x) = -(x^2) + 4$, then by the power rule for anti-derivatives, $F(x) = -\frac{1}{3}x^3 + 4x$.

$F(2) - F(-2) = -\frac{8}{3} + 8 - \frac{8}{3} + 8 = 16 - \frac{16}{3} = \frac{32}{3}$.

5. Find the volume of the solid constructed by rotating the above region about the x-axis.

(*) We derived a very handy formula in class for this.
The volume is given by $V = \int_a^b \pi(f(x))^2 dx$.

From the last problem I know the limits of integration are ± 2. And π is a constant, so I can pull it out of the integral. So I have:

$$\begin{aligned} V &= \pi \int_{-2}^{2} (-(x^2) + 4)^2 dx \\ &= \pi \int_{-2}^{2} x^4 - 8x^2 + 16 dx \\ &= \pi(\tfrac{1}{5}x^5 - \tfrac{8}{3}x^3 + 16x)|_{-2}^{2} \\ &= \pi(\tfrac{1}{5}(32) - \tfrac{8}{3}(8) + 16(2) - (\tfrac{1}{5}(-32) - \tfrac{8}{3}(-8) + 16(-2))) \\ &= \tfrac{512}{15}\pi \end{aligned}$$

6. Find the area of the region bounded by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

(*) First I need to know my limits of integration. I know the region is bounded by the points where the two curves meet, so I will find their intersections. If $\sqrt{x} = x^2$, then clearly $x = 0$ is a solution. Also, by squaring both sides I get $x = x^4$. Then $1 = x^3$. So $x = 1$ is a solution. So I know the limits of integration are 0 and 1.

Next I observe that on the interval $[0, 1]$, the curve $f(x) = \sqrt{x}$ is the higher curve, and $g(x) = x^2$ is the lower curve. So the area is given by $\int_0^1 f(x) - g(x) dx$.

$$\begin{aligned} A &= \int_0^1 f(x) - g(x) dx \\ A &= \int_0^1 \sqrt{x} - x^2 dx \\ A &= (\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{3}x^3)|_0^1 \\ A &= (\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{3}x^3)|_0^1 \\ A &= (\tfrac{2}{3}(1)^{\frac{3}{2}} - \tfrac{1}{3}(1)^3) \\ A &= \tfrac{1}{3} \end{aligned}$$

7. Consider the following supply function, $S(x)$, and demand function, $D(x)$, where price is a function of quantity.

$$S(x) = x^2 - 2x + 2, D(x) = -2x + 11$$

a. Find the equillibrium point (the value of x such that supply equals demand).

(*) This is x such that $S(x) = D(x)$. So I solve $x^2 - 2x + 2 = -2x + 11$ to get $x^2 = 9$, or $x = \pm 3$. Now x can't be negative (by a restricted domain argument with respect to the application), so I only consider the case when $x = 3$. When $x = 3$, then $S(3) = D(3) = 5$. The equillibrium point is $(3, 5)$.

b. Find the consumer's surplus at this point.

(*) The consumer's surplus is given by $\int_0^Q D(x)dx - QP$ where Q is the quantity and P is the price.

First, $Q = x = 3$ at the equilibrium point. So $P = D(3) = 5$. Then I have:

$$\begin{aligned} \text{surplus} &= \int_0^Q D(x)dx - QP \\ &= \int_0^3 -2x + 11dx - (3)(5) \\ &= (-x^2 + 11x)|_0^3 - 15 \\ &= (-9 + 33) - 15 \\ &= 9 \end{aligned}$$

c. Find the producer's surplus at this point.

(*) The producer's surplus is given by $QP - \int_0^Q S(x)dx$ where Q is the quantity and P is the price.

Recall $Q = x = 3$ at the equilibrium point. Alos, $P = S(3) = 5$. Then I have:

$$\begin{aligned} \text{surplus} &= QP - \int_0^Q S(x)dx \\ &= (3)(5) - \int_0^3 x^2 - 2x + 2dx \\ &= 15 - (\tfrac{1}{3}x^3 - x^2 + 2x)|_0^3 \\ &= 15 - 6 \\ &= 9 \end{aligned}$$

8. Find the area under the piece-wise defined function from $x = 0$ to $x = 99$.

$$f(x) = \begin{cases} 1 & \text{if } x \le 50; \\ 2 & \text{if } x > 50. \end{cases}$$

(*) I start by attempting to compute $\int_0^{99} f(x)dx$. I have one complication though: $f(x)$ has two meanings, depending on the value of x. But I know I can "split" the integral. So I will do that in such a way that it is clear what I need to anti-differentiate.

First, $\int_0^{99} f(x)dx = \int_0^{50} f(x)dx + \int_{50}^{99} f(x)dx$. For the left integral it is clear I should use "1" for $f(x)$, as x is always less than or equal to 50. For the right integral it is clear I should use "2" for $f(x)$, as x is always greater than 50. So I have:

$$\begin{aligned} \int_0^{99} f(x)dx &= \int_0^{50} f(x)dx + \int_{50}^{99} f(x)dx \\ &= \int_0^{50} 1dx + \int_{50}^{99} 2dx \\ &= (x)|_0^{50} + (2x)|_{50}^{99} \\ &= (50) + (198 - 100) \\ &= 148 \end{aligned}$$

9. Find the amount of continuous money flow in which 5000 per year is being invested at 6%, compounded continuously, for 45 years.

(*) I can compute the amount of continuous money flow by calculating $\int_0^{45} 5000e^{.06t}dt$. Do you remember why that is?

$$\begin{aligned} \text{funds} &= \int_0^{45} 5000e^{.06t}dt \\ &= 5000 \int_0^{45} e^{.06t}dt \\ &= 5000(\tfrac{1}{.06}e^{.06(45)} - \tfrac{1}{.06}e^{.06(0)}) \\ &= 5000(\tfrac{1}{.06}e^{.06t})|_0^{45} \\ &\approx 1,156,644.31 \end{aligned}$$

I'm rich!

10. In 1897 the world's consumption of Watsonium-199 was $55,000$ ft^3. The amount used has steadily increased at an average rate of 10% per year. What is the consumption rate in 2007? What was the total amount of Watsonium-199 consumed?

(*) First I need to establish my model. Clearly we have an exponential growth situation, so I use $w(t) = Ce^{kt}$, where $w(t)$ is the amount of Watsonium-199 consumed at year t.

C, the initial amount, is 55000. k is .10. So I have $w(t) = 55000e^{.1t}$.

Then $w(110)$ is the amount consumed 110 years since 1897, or in 2007. $w(110) \approx 3,293,077,794$.

The total amount consumed in the 110 year time span is $\int_0^{110} w(t)dt$. This is given as:

$$\begin{aligned} \text{total} &= \int_0^{110} w(t)dt \\ &= \int_0^{110} 55000e^{.1t}dt \\ &= 55000\int_0^{110} e^{.1t}dt \\ &= 55000(\tfrac{1}{.1}e^{.1t})|_0^{110} \\ &= 55000(\tfrac{1}{.1}e^{.1(110)} - \tfrac{1}{.1}e^{.1(0)}) \\ &\approx 32,930,227,943 \end{aligned}$$

11. Find $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$ where $f(x) = 2x$ on the closed interval $[3,5]$.

(*) This is the limit definition of the definite integral, equivalent to $\int_3^5 2xdx$. The antiderivative of $f(x)$, which we'll call $F(x)$, is x^2. Then $F(5) - F(3) = 25 - 9 = 16$.

12. (40) Compute each of the following:

a. $\int 6x^3 + 6\sqrt{x}dx$

(*) Use the power rule. $\frac{3}{2}x^4 + 4x^{\frac{3}{2}} + C$

b. $\int \frac{5}{x}dx$

(*) Note $\int \frac{5}{x}dx = 5\int \frac{1}{x}dx$. Then we have $5\ln|x| + C$.

c. $\int e^x + 2e^x dx$

(*) The antiderivative of e^x is e^x. We have $e^x + 2e^x + C$.

d. $\int \frac{6x^2+5}{2x^3+5x+1}dx$

(*) Use u-substitution here. Let $u = 2x^3 + 5x + 1$. We get $\ln|2x^3 + 5x + 1| + C$.

e. $\int 6xe^{x^2+1}dx$

(*) Use u-substitution again, with $u = x^2 + 1$. We get $3e^{1+x^2} + C$

f. $\int 15x^2(x^3+2)^9 dx$

> **(*)** We have to use u-substitution again. $u = x^3 + 2$. Then we have $\frac{1}{2}(x^3+2)^{10} + C$.

g. $\int_0^1 x^2 + x dx$

> **(*)** Let $f(x) = x^2 + x$. Then $F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2$. $F(1) - F(0) = \frac{5}{6}$.

h. $\int_1^2 \frac{x-1}{x^2-2x} dx$

> **(*)** We have $f(x) = \frac{x-1}{x^2-2x}$. To find $F(x)$ we need to use u-substitution. Let $u = x^2 - 2x$. Then we get $F(x) = \frac{1}{2}\ln(|x^2 - 2x|)$. Note that $F(2)$ is not a real number! (Because we'd take the log of 0). So $F(2) - F(1)$ is not a real number.

13. (20) Watsonovia Bank offers an account which pays interest at the rate of 6%, compounded continuously.

a. I deposit 10,000 dollars into an account. What is its value after 20 years?

> **(*)** This is a precalculus problem. My balance is given by $P(t) = Pe^{rt}$, with $P = 10000$, $r = .06$, and $t = 20$. The value is 33201.20.

b. Find the future value of a continous money flow of 1,000 dollars per year for 40 years.

> **(*)** Now I consider the case that I make continuous deposits at the rate of 1000 per year. I want to find the accumulated total over 40 years. I set up the definite integral $\text{TOTAL} = \int_0^{40} 1000e^{.06t} dt$.
>
> If $f(t) = 1000e^{.06t}$, then $F(t)$ is computed by dividing by r. So I get $F(t) = 16666.67e^{.06t}$.
>
> $F(40) - F(0) = 167053$.