1. Differentiate.

a. $f(x) = (x^2 + 6x)^5 (3x^5 + 9)^4$

(*) Use the product rule with $(x^2 + 6x)^5$ as one function, and $(3x^5 + 9)^4$ as the other.

$f'(x) = 60x^4 \left(3x^5 + 9\right)^3 \left(x^2 + 6x\right)^5 + 5(2x + 6) \left(3x^5 + 9\right)^4 \left(x^2 + 6x\right)^4$

b. $f(x) = \ln(\frac{x^2-1}{x+1})$

(*) There are three ways to take this. You can find the derivative of $\frac{x^2-1}{x+1}$ and divide that derivative by $\frac{x^2-1}{x+1}$ itself. You can use the properties of logarithms to write $f(x)$ as $\ln(x^2 - 1) - \ln(x + 1)$ and differentiate both pieces. You can also note $\frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x - 1$. Then $f(x) = \ln(x - 1)$ and $f'(x) = \frac{\frac{d}{dx}(x-1)}{x-1} = \frac{1}{x-1}$.

c. $f(x) = \ln(\frac{1}{\sqrt{3x+2}})$.

(*) Note $\frac{1}{\sqrt{3x+2}} = (3x + 2)^{-\frac{1}{2}}$.

$f'(x) = -\frac{3}{2(3x+2)}$

d. $f(x) = \frac{x^2+19x+1}{x+1}$

(*) You have to use the quotient rule here.

$f'(x) = \frac{2x+19}{x+1} - \frac{x^2+19x+1}{(x+1)^2}$

e. $f(x) = e^{x^2+6x}$

(*) The long way to do this is to use the chain rule (with the "inside function" being the exponent). You can also apply the shortcut: multiply $f(x)$ by the derivative of the exponent. The derivative of the exponent is $2x + 6$. So I get

$f'(x) = (2x + 6)e^{x^2+6x}$.

f. $f(x) = (x^2+4)^5 + (x^2+4)^4$

> **(*)** Use the chain rule or general power rule for both $(x^2+4)^5$ and $(x^2+4)^4$.
> $f'(x) = 10x\left(x^2+4\right)^4 + 8x\left(x^2+4\right)^3$.

g. $f(x) = x\ln(x)$

> **(*)** You have to use the product rule here, with the first function being x and the second being $\ln(x)$.
>
> $f'(x) = \ln(x) + 1$.

h. $f(x) = \ln(\frac{(x^2+4)^2(x+3)}{(x-5)e^x})$. *Hint. Think about the properties of logarithms and use them to write a "nicer" function.*

> **(*)** This will get very ugly if you ignore the hint (but it can be done). Use the properties of logarithms to write
>
> $f(x) = \ln(\frac{(x^2+4)^2(x+3)}{(x-5)e^x}) = 2\ln(x^2+4)) + \ln(x+3) - \ln(x-5) - \ln(e^x)$.
>
> Then $f'(x) = 2\frac{2x}{x^2+4} + \frac{1}{x+3} - \frac{1}{x-5} - 1$

i. $f(x) = x^2e^{2x}$

> **(*)** You have to use the product rule here. One function is x^2, and the other is e^{2x}.
>
> $f'(x) = 2e^{2x}x^2 + 2e^{2x}x$

2. Consider the function $f(x) = x^3 + 3x^2 + 3x + 1$. Find the *absolute* max and *absolute* minimum over the domain $[-4, 4]$.

> **(*)** First solve $f'(x) = 0$ for x to find the relative extrema.
>
> $f'(x) = 3x^2 + 6x + 3$.
> $f'(x) = 0$ has solution $x = -1$.
>
> Now I know the min or max can occur at $x = -1$, but I also need to check the closed endpoints ($x = -4$ and $x = 4$).
>
> When $x = 4$, then $f(x) = 125$. When $x = -1$ then $f(x) = 0$. When $x = -4$ then $x = -27$. So the absolute min is $(-4, -27)$ and the absolute max is $(4, 125)$.

3. I wish to enclose a rectangular lot with fencing. The area of the lot must be 400 square feet. One side of the lot borders a river. Find the dimensions of the lot which minimize the amount of fencing required.

(*) The amount of fencing required is given by $A = 2l + w$.

I have the constraint $lw = 400$.

Dividing by l I get $w = \frac{400}{l}$.

Then substituting back into my fencing function, I have A as a function of l:
$A(l) = 2l + \frac{400}{l}$.

Now $A'(l) = 2 - \frac{400}{l^2}$.

Solving $A'(l) = 2 - \frac{400}{l^2} = 0$, I first add the fraction to both sides: $2 = \frac{400}{l^2}$.

Then I can multiply by l^2: $2l^2 = 400$.

Dividing by 2: $l^2 = 200$.

Then $l = \sqrt{200}$.

I can plug l into the constraint to get $w = \frac{400}{\sqrt{200}}$.

4. A craftsman wants to make a cylindrical jewelry box that has volume, V, equal to 100 cubic inches. He will make the base and side of the box out of a metal that costs 90 cents per square inch. The lid of the box will be made from a metal with a more ornate finish which costs 700 cents per square inch. Find the value of r, the radius of the base, for which we have a potential relative extreme point of C, the total cost of the materials. Is the extreme point a minimum or a maximum? How do you know?

(*) I want to optimize cost. The cost is 90 cents per square inch of the base, or $90\pi r^2$. The cost of the lid is 700 cents, or $700\pi r^2$. The cost of the side is 90 cents per square inch, or $180\pi r h$. Then the total cost is $C = 790\pi r^2 + 180\pi r h$.

I have that $V = \pi r^2 h = 100$. So I can solve for h to get $h = \frac{100}{\pi r^2}$. Then I substitute h in to the function C to get

$C = 790\pi r^2 + 180\pi r \frac{100}{\pi r^2}$

$C = 790\pi r^2 + \frac{18000}{r}$

$C = 790\pi r^2 + 18000 r^{-1}$

$C' = 1580\pi r - 18000 r^{-2}$

Solve $C' = 0$ for r: $C' = 1580\pi r - \frac{18000}{r^2} = 0$.

$1580\pi r = \frac{18000}{r^2}$

$1580\pi = \frac{18000}{r}$

$r \approx 3.62632$.

This value of r minimizes C. To verify, note $C'' = 1580\pi + 18000 r^{-3}$. When I plug in my value for r, C is positive. This means C is concave up at $r = 3.62632$, so I have a maximum.

5. A manufacturer wants to design an open-top box having a square base and a surface area of 216 square inches. What dimensions will provide a box with maximum volume?

(*) With a square base, the volume is given by $V = l^2h$. (length = width).

I'm told the surface area, or sum of the areas of the five faces (remember, there is no top) is 216. This gives me

$SA = 216 = l^2 + 4lh$. (The base and four sides)

I can solve for h to get $h = \frac{216-l^2}{4l}$.

Then I subsitute this expression for h in the volume. I get $V = l^2\frac{216-l^2}{4l} = 4l(216 - l^2) = 1064l - 4l^3$.

I find the derivative $V' = 1064 - 12l^2$.

Then I solve $V' = 0$ for l.

$1064 - 12l^2 = 0$

$1064 = 12l^2$

$\frac{266}{3} = l^2$

$9.4163 = l$.

Since the length is 9.4163, then $h = \frac{216-l^2}{4l} = 3.38066$

6. Let $f(5) = 12, g(5) = 10, f'(5) = -3, g'(5) = -8$. Find $h'(5)$ if $h(x) = xe^{f(x)}$.

(*) I have to use the product rule here. One function is x. Its derivative is 1. The other function is $e^{f(x)}$. Using the shortcut to the chain rule described in problem 1, I can quickly find that its derivative is $f'(x)e^{f(x)}$. So I have

$h'(x) = e^{f(x)} + xf'(x)e^{f(x)}$.

Let $x = 5$ to get $h'(5) = -15e^{12}$.

7. Let $f(5) = 12, g(5) = 10, f'(5) = -3, g'(5) = -8$. Find $h'(5)$ if $h(x) = \sqrt{f(x)g(x)}$.

(*) First write $h(x) = (f(x)g(x))^{\frac{1}{2}}$.

Now use the general power rule to write $h'(x) = \frac{1}{2}(f(x)g(x))^{\frac{-1}{2}}\langle\text{derivative of } f(x)g(x)\rangle$.

To find the derivative of $f(x)g(x)$, use the product rule. It is $f'(x)g(x) + f(x)g'(x)$. Then

$h'(x) = \frac{1}{2}(f(x)g(x))^{\frac{-1}{2}}(f'(x)g(x) + f(x)g'(x))$.

Plug in 5 for x to get $h'(5) = \frac{1}{2}(120)^{\frac{-1}{2}}(-126)$

8. A spherical snowball is melting. The measure of its diameter is decreasing at the rate of 4 inches per hour. At what rate is the volume decreasing when the diameter is 6 inches?

(*) The volume of a sphere is $V = \frac{4}{3}\pi r^3$. But the radius is a function of time. So we have

$V = \frac{4}{3}\pi(r(t))^3$.

Then $V' = 4\pi(r(t))^2 r'(t)$.

We're told the radius, $r(t)$ is 6. We're then told the rate of change of radius, $r'(t)$, is -4. So

$V' = 4\pi 6^2(-4) = -576$.

9. In October 2009 Milton's consumption of cat food iss $20\mathrm{lb}$ per month, and is increasing at the rate of $.5\mathrm{lb}$ per month. The cost per pound is 2 dollars, and is increasing at the rate of .05 dollars per month. What is Milton's total food bill for the month? At what rate is it increasing?

(*) Let $f(t)$ represent Milton's consumption in pounds, and $g(t)$ represent the price of food per pound. Then the total bill is $T(t) = f(t)g(t)$.

I want to find the *rate of change* of the food bill, or $T'(t)$.

To do so, I need to use the product rule. $T'(t) = \frac{d}{dt}f(t)g(t) = f'(t)g(t) + f(t)g'(t)$

$f'(t)$ is the rate of change of consumption, which is .5. $g'(t)$ is the rate of change of price, which is .05.

Then the rate of change of the bill is $f'(t)g(t) + f(t)g'(t) = (.5)(2) + (20)(.05) = 2$

10. Let $f(x) = 4^x$. Let $g(x) = e^{kx}$. Find the value of k so that $f(x) = g(x)$.

(*) Set up the equation $4^x = e^{kx}$

To get the x out of the exponent, take the log of both sides: $\ln(4^x) = \ln(e^{kx})$.

Then $x\ln(4) = kx\ln(e)$

Then $x\ln(4) = kx$

Then $k = \ln(4)$.

11. Solve for x: $\ln(\ln(x)) = 0$.

(*) $\ln(\ln(x)) = 0$

$e^{\ln(\ln(x))} = e^0$

$\ln(x) = 1$

$e^{\ln(x)} = e^1$

$x = e$

12. Milton City boasted a total population of 90,000 in 1990. In 2000, the population had increased to 100,000. Assuming exponential growth, predict the population in the year 2050. Find the rate at which the population is increasing at that time.

(*) The city population is given by the exponential model $P(t) = P_0 e^{kt}$.

If I let t be years since 1990, then I know first that $P_0 = 90000$.

So I have $P(t) = 90000e^{kt}$.

When $t = 10$ then $P(t) = 100000$, so $100000 = 90000e^{10k}$.

Then $\frac{10}{9} = e^{10k}$.

Taking $\ln()$ of both sides: $\ln(\frac{10}{9}) = 10k$.

Hence, $k = \frac{1}{10}\ln(\frac{10}{9})$.

Now I have $P(t) = 90000e^{\frac{1}{10}\ln(\frac{10}{9})t}$, so in 2050 the population is:

$P(60) = 90000e^{\frac{1}{10}\ln(\frac{10}{9})(60)}$.

13. Scientists unearth the legendary tomb of the Cat god Milton, lord of hairballs, and wish to determine its age using Carbon dating. They determine 31% of the tombs' original amount of Carbon-14 remains. Assuming Carbon-14 has a half-life of 5750 years, and the **exponential decay** relation $A(t) = A_0 e^{-kt}$,

a. Find the value of the unknown parameter k, where $A(t)$ is the current amount of Carbon-14 present, and t is the number of years that have passed.

(*) I solve for k, $\frac{1}{2}A_0 = A_0 e^{-5750k}$

Dividing by A_0 I see $\frac{1}{2} = e^{-5750k}$

Applying $\ln()$ to both sides: $\ln(\frac{1}{2}) = -5750k$

Then $k = -\frac{\ln(\frac{1}{2})}{5750}$

b. Determine the age of the tomb.

(*) I have the model $A(t) = A_0 e^{\frac{\ln(\frac{1}{2})}{5750}t}$.

I want to find t when $A(t) = .31A_0$.

So I solve for t in $.31A_0 = A_0 e^{\frac{\ln(\frac{1}{2})}{5750}t}$.

Dividing by A_0: $.31 = e^{\frac{\ln(\frac{1}{2})}{5750}t}$.

Applying $\ln()$ to both sides: $\ln(.31) = \frac{\ln(\frac{1}{2})}{5750}t$.

Then dividing by $\frac{\ln(\frac{1}{2})}{5750}$ I see: $t = \frac{\ln(.31)}{\frac{\ln(\frac{1}{2})}{5750}}$.

14. The number of bacteria in a flask grows according to the differential equation $\frac{dy}{dt} = 0.08y$, where time, t is measured in hours. The initial number of bacteria is 500,000. Find a formula for $y(t)$, the number of bacteria at time t. Use your formula to predict the number of hours to pass before the number of bacteria doubles.

(*) Because I have the relationship $\frac{dy}{dt} = 0.08y$, then the function y must have the form $y = Ce^{kt}$. I'm told the initial amount, C, is 500,000, and that $k = 0.08$. Then I know $y = 500000e^{.08t}$.

I want to find t when y is twice the initial amount, or 1,000,000. So I set up and solve the equation

$1000000 = 500000e^{.08t}$.

$2 = e^{.08t}$

$\ln(2) = \ln(e^{.08t})$

$\ln(2) =, 08t$

$t = \frac{\ln(2)}{.08}$.