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You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots$$

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1. [11 points] Indicate if each of the following is true or false by circling the correct answer. Justify your answer.

a. [2 points] If the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ is 2, then $\sum_{n=0}^{\infty} a_n$ diverges.

True **[False]**

Solution: $x = 5$ belongs to the interval of convergence then $\sum_{n=0}^{\infty} a_n$ converges.

b. [2 points] If $P(x)$ is a cumulative distribution function with $P(0) = \frac{1}{3}$, then the median is positive.

[True] False

Solution:
Since $P(x)$ is increasing and $P(median) = \frac{1}{2}$ then median> 0.

c. [3 points]
If $F(x) = \int_{-x^2}^{0} \frac{1}{1+t^4} dt$ then $F(x)$ is decreasing for $x > 0$.

True **[False]**

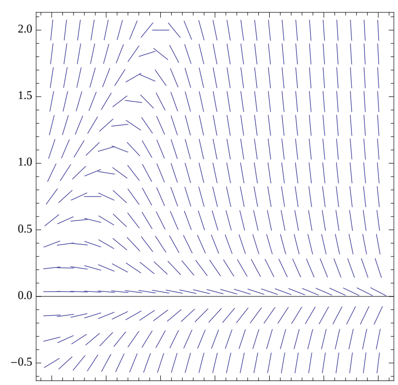
Solution: $F'(x) = \frac{2x}{1+x^8} > 0$ for $x > 0$. Hence $F(x)$ is increasing for $x > 0$.

d. [2 points] The differential equation $y' = (y - x^3)y$ has two equilibrium solutions, $y = 0$ and $y = x^3$.

True **[False]**

Solution: $y = x^3$ is not an equilibrium solution.

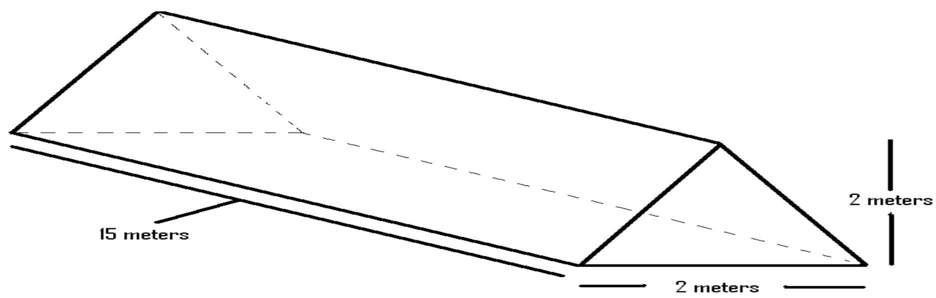
e. [2 points] Using the slope field below, we can guarantee that the solution with initial condition $y(0) = \frac{1}{2}$ satisfies $y(3) < 0$.



True **[False]**

Solution: $y' > 0$ for $y < 0$, hence $y(3) \geq 0$.

2. [7 points] Deep beneath Dennison Hall lies a large septic tank. It has the shape of a triangular prism with the dimensions depicted below.



Suppose that the tank described above is full of sewage and that this sewage has a density of $1000(1+e^{-2x})\ \frac{kg}{m^3}$, where x is the distance in meters above the base of the tank.

a. [5 points] Find a definite integral that computes the mass of the sewage in the tank.

Solution:
Slice dimensions: $2-x$ by 15 by Δx.
Hence

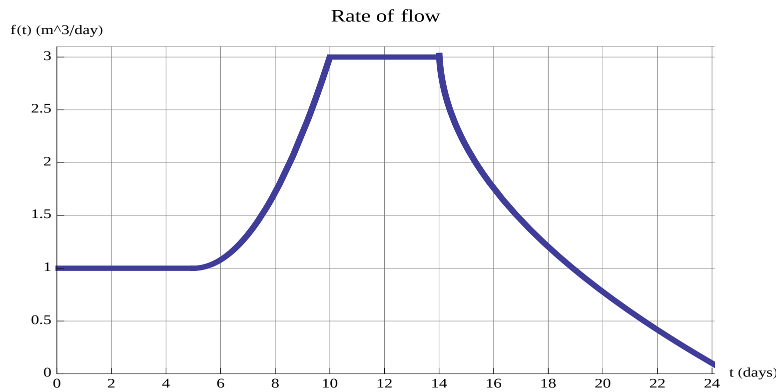
$$m = \int_0^2 \delta(x)(2-x)(15)dx = \int_0^2 1000(1+e^{-2x})(2-x)(15)dx$$

b. [2 points] Compute the value of the integral using your calculator. Do not forget to include the units.

Solution:

$$m = \int_0^2 1000(1+e^{-2x})(2-x)(15)dx = 41318.78\text{kg}$$

3. [11 points] Sewage flows into the tank described in the previous problem at a rate of $f(t)$ cubic meters per day. Let t be the number of days since December 1, when the tank had 1 m^3 of sewage. A graph of $f(t)$ is given below. Use it to answer the following questions.



a. [3 points] Suppose that $V(t)$ gives the volume of sewage in the tank at time t. Find a formula for $V(t)$ in terms of $f(t)$.

Solution: $V(t) = 1 + \int_0^t f(t)dt$

b. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave up? ________________

Solution: $4 < t < 10$

c. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave down? ________________

Solution: $14 < t < 24$

d. [4 points] Fill out the table below. Using the values in your table, compute Riemann sums with 3 subintervals to find an underestimate and an overestimate for $V(12)$. Justify why the Riemann sums you selected yield the appropriate under and upper estimates. Do not forget to include the units in your answer.

t	0	4	8	12
$f(t)$				

Solution:

t	0	4	8	12
$f(t)$	1	1	$\approx$ 1.75	3

$f(t)$ is increasing in $(0, 12)$ then
Upper estimate (Right hand sum): $4(1 + 1.75 + 3) + 1 = 24\ m^3$
Lower estimate (Left hand sum): $4(1 + 1 + 1.75) + 1 = 16\ m^3$

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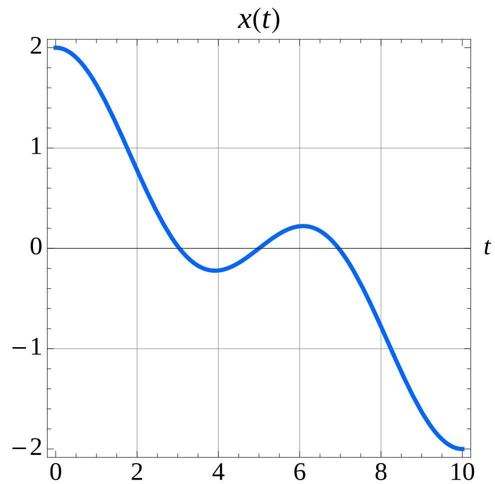
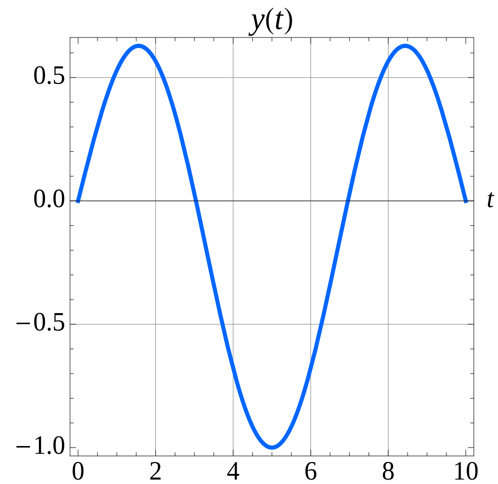
4. [11 points] A particle is moving in the x-y plane according to the parametric equations $(x(t), y(t))$ for $0 \leq t \leq 10$. The graph of these functions are shown below.





a. [2 points] What are the starting and ending points of the particle?

Solution: Starting point $(2, 0)$ and ending point $(-2, 0)$.

b. [3 points] At which values of $0 < t < 10$ is the particle moving horizontally straight to the right or to the left?

Solution: Right: $t = 5$.
Left: $t = 2$ and $t = 8.5$.

c. [2 points] At which values of $0 < t < 10$ is the particle moving straight up or down?

Solution: Up: $t = 6$
Down: $t = 4$.

d. [2 points] At which values of t is the particle at the origin?

Solution: $t = 3$ and $t = 7$.

e. [2 points] If $v(t)$ is the speed of the particle at time t, which one is larger $v(2)$ or $v(5)$? Explain.

Solution: $v(5) < v(2)$ since the horizontal and vertical velocities satisfy $v_x(5)^2 < v_x(2)^2$ and $v_y(5)^2 < v_y(2)^2$.

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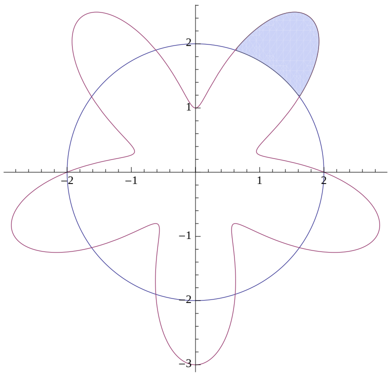
5. [9 points]

a. [3 points] In the picture below, the graphs of $r = 2$ and $r = 2 - \sin(5\theta)$ are shown. Write a definite integral that computes the shaded area.



Solution: Endpoints satisfy: $2 = 2 - \sin(5\theta)$, hence $\sin(5\theta) = 0$. Then the endpoints are $\theta = \frac{\pi}{5}, \frac{2\pi}{5} = .628, 1.25$ radians.

$$A = \frac{1}{2}\int_{\frac{\pi}{5}}^{\frac{2\pi}{5}} (2 - \sin(5\theta))^2 - 4 d\theta$$

b. [6 points] Find parametric equations of the tangent line to the limacon $r = \sin(\theta) - \frac{1}{2}$ at $\theta = \frac{\pi}{4}$.

Solution:

$$x(\theta) = (\sin\theta - \frac{1}{2})\cos\theta \qquad x(\frac{\pi}{4}) = .146$$

$$x'(\theta) = (\sin\theta - \frac{1}{2})(-\sin\theta) + \cos^2(\theta) \qquad x'(\frac{\pi}{4}) = .353$$

$$y(\theta) = (\sin\theta - \frac{1}{2})\sin\theta \qquad y(\frac{\pi}{4}) = .146$$

$$y'(\theta) = (\sin\theta - \frac{1}{2})\cos\theta + \sin\theta\cos\theta \qquad y'(\frac{\pi}{4}) = .646$$

$$x_{tan}(t) = .146 + .353\ t$$

$$y_{tan}(t) = .146 + .646\ t$$

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6. [8 points] Let $P(t)$ be the population of birds living in a lake t days after January 1, 2009. It has been noticed that the rate of growth of the population of birds varies depending on the season of the year. To take this into consideration, we assume the rate of growth of the population is equal to $k(t)P$, where $k(t) = \frac{1}{100}\sin(\frac{2\pi}{365}t)$. Hence $P(t)$ satisfies

$$\frac{dP}{dt} = k(t)P$$

There were 100 birds living in the lake in January 1, 2009.

a. [6 points] Solve the differential equation satisfied by $P(t)$ and find the population of birds after 100 days after January 1, 2009.

Solution:

$$\frac{dP}{dt} = \frac{1}{100}\sin\left(\frac{2\pi}{365}t\right)P$$
$$\frac{dP}{P} = \frac{1}{100}\sin\left(\frac{2\pi}{365}t\right)dt$$
$$\ln|P(t)| = -\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}t\right) + C$$
$$P(t) = D\ e^{-\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}\ t\right)}$$

Using the initial condition $P(0) = 100$, we get

$$100 = D\ e^{-\frac{365}{100(2\pi)}}$$
$$D = 100\ e^{\frac{365}{100(2\pi)}} = 178.767.$$

$P(t) = 178.767\ e^{-\frac{365}{100(2\pi)}\cos\left(\frac{2\pi}{365}\ t\right)}$ and $P(100) \approx 195$ birds.

b. [2 points] Graph the solution $P(t)$ in your calculator and use this graph to answer the following questions:

1. What is the maximum and minimum amount of birds living in the lake throughout the year?
2. When are the maximum and minimum expected to occur?

Solution: Maximum at $t = \frac{365}{2} = 182.5$ days there are 319 birds.
Minimum at $t = 0$ or $t = 365$ there are 100 birds,

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7. [14 points] For each of the following sequences

1.Compute $\lim_{n\to\infty} a_n$.

2.Decide if $\sum_{n=0}^{\infty} a_n$ converges or diverges. Circle your answer.

Support your answer by stating the test(s) or facts you used to prove convergence or divergence , and show complete work and justification.

a. [4 points]

$$a_n = \left(\frac{-1}{\pi}\right)^n \qquad \lim_{n\to\infty} a_n = \underline{\hspace{2cm}} \qquad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

Solution: Sequence: $a_n = r^n$ where $|r| = \left|\frac{-1}{\pi}\right| = .318 < 1$ hence $\lim_{n\to\infty} a_n = 0$.

Series: $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} r^n$ is a geometric series with $|r| < 1$ then it **converges**.

b. [4 points]

$$a_n = \frac{n^2+2}{1+4n^2} \qquad \lim_{n\to\infty} a_n = \underline{\hspace{2cm}} \qquad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

Solution: Sequence: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2}{1+4n^2} = \lim_{n\to\infty} \frac{n^2}{4n^2} = \frac{1}{4}$.

Series: Since a_n does not converge to 0 then $\sum_{n=0}^{\infty} a_n$ diverges.

Note to the graders: The criteria for the justification of the divergence of the series has been given different names in some sections (nth term test and some others). If you see these kind of justifications, please ask the instructor before considering any deductions.

c. [6 points]

$$a_n = \frac{n}{\sqrt{n^4+5}} \qquad \lim_{n\to\infty} a_n = \underline{\hspace{2cm}} \qquad \sum_{n=0}^{\infty} a_n : \quad \text{Converges} \quad \text{Diverges}$$

Solution: Sequence: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^4+5}} = \lim_{n\to\infty} \frac{n}{n^2} = 0$.

Series: $a_n = \frac{n}{\sqrt{n^4+5}} \sim \frac{1}{n}$ as n goes to infinity.
Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^4+5}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+5}} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

$\sum_{n=0}^{\infty} a_n \sim \sum_{n=0}^{\infty} \frac{1}{n}$. Hence diverges.

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8. [14 points] Consider the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n(2n+1)}(x-3)^n$$

a. [11 points] For what values of x does the power series converge?

Solution: Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{4^{n+1}(2n+3)}(x-3)^{n+1}\right|}{\left|\frac{(-1)^n}{4^n(2n+1)}(x-3)^n\right|} = |x-3| \lim_{n\to\infty} \frac{2n+1}{4(2n+3)}$$

$$= |x-3| \lim_{n\to\infty} \frac{2n}{8n} = \frac{|x-3|}{4}$$

Then the series converge for values of x satisfying $\frac{|x-3|}{4} < 1$. Then the series converge for $-1 < x < 7$.

Enpoints ($x = -1, 7$): $x = -1$ yields $\sum_{n=1}^{\infty} \frac{1}{2n+1}$. Limit Comparison Test with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} == \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

$x = 7$ yields $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$. Alternating series test: $a_n = f(n) = \frac{1}{2n+1}$

- $\lim_{n\to\infty} \frac{1}{2n+1} = 0$.
- a_n decreasing: $f'(n) = \frac{-2}{(2n+1)^2} < 0$ for $n > 0$.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges. Hence the interval of convergence of the series is $(-1, 7]$.

b. [2 points] For what values of x does the power series converges absolutely?

Solution: $-1 < x < 7$

c. [1 point] For what values of x does the power series converges conditionally?

Solution: $x = 7$

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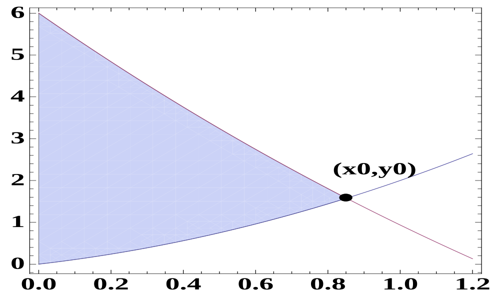
9. [15 points] The graph shows the area between the graphs of $f(x) = 6\cos(\sqrt{2x})$ and $g(x) = x^2 + x$. Let (x_0, y_0) be the intersection point between the graphs of $f(x)$ and $g(x)$.



a. [6 points] Compute $P(x)$, the function containing the first three nonzero terms of the Taylor series about $x = 0$ of $f(x) = 6\cos(\sqrt{2x})$.

Solution: $P(x) = 6\left(1 - \frac{(\sqrt{2x})^2}{2!} + \frac{(\sqrt{2x})^4}{4!}\right) = 6\left(1 - x + \frac{x^2}{3!}\right) = 6 - 6x + x^2$

b. [3 points] Use $P(x)$ to approximate the value of x_0.

Solution: To approximate x_0 you need to solve $P(x) = x + x^2$. Then $6 - 6x + x^2 = x + x^2$ implies $x_0 = \frac{6}{7} = .857$.

c. [3 points] Use $P(x)$ and the value of x_0 you computed in the previous question to write an integral that approximates the value of the shaded area. Find the value of this integral.

Solution: $A = \int_0^{x_0} P(x) - (x + x^2)dx = \int_0^{\frac{6}{7}} 6 - 7x \; dx = 2.571$

d. [1 point] Graph $f(x)$ and $g(x)$ in your calculator. Use the graphs to find an approximate value for x_0.

Solution: $x_0 \approx .851$

e. [2 points] Write a definite integral in terms of $f(x)$ and $g(x)$ that represents the value of the shaded area. Find its value using your calculator.

Solution: $A = \int_0^{.851} 6\cos(\sqrt{2x}) - (x + x^2) \; dx = 2.562$