You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots\cdots \qquad \text{for all values of } x$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

1. [10 points] For each question, circle if the statement is always true or false. **No justification is necessary**.

a. [2 points] If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

True $\boxed{\text{False}}$

Solution: Example: If $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but $\sum_{n=1}^{\infty} b_n$ diverges.

b. [2 points] If $F(x) = \int_{1-x}^{x^3} e^{-t^4} dt$ then $F(x)$ is increasing.

$\boxed{\text{True}}$ False

Solution: $F'(x) = 3x^2 e^{-x^{12}} + e^{-(1-x)^4} > 0$ then $F(x)$ is increasing.

c. [2 points]

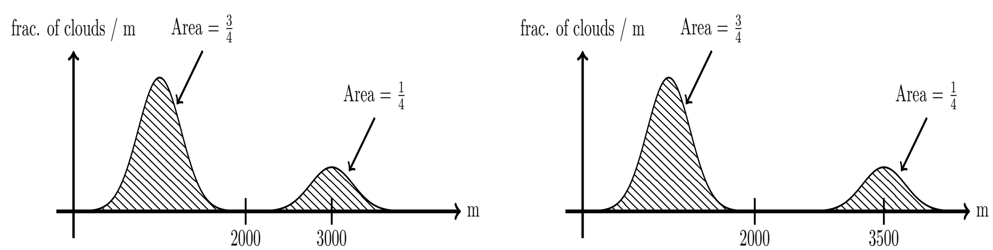
$$\ln(2.5) = 1.5 - \frac{1}{2}(1.5)^2 + \frac{1}{3}(1.5)^3 - \frac{1}{4}(1.5)^4 + \cdots$$

True $\boxed{\text{False}}$

Solution: The Taylor series for $\ln(1+x)$ is not valid for $x = 1.5$.

d. [2 points] The left graph is a probability density function for the height of clouds in the sky. A gust of wind causes all clouds higher than 2000 meters to rise an additional 500 meters. The right graph shows the heights of clouds afterwards. The median cloud height after the wind is 500/4 meters higher than the median cloud height before the wind.

True $\boxed{\text{False}}$

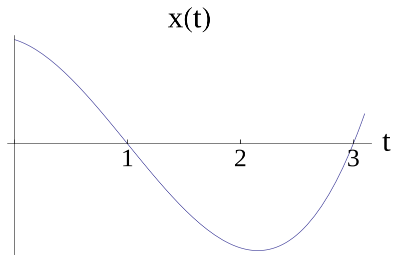


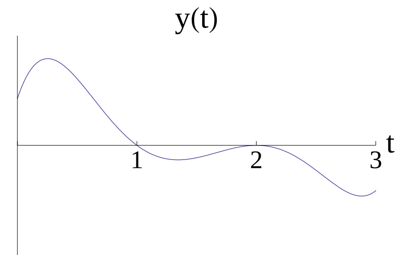
Solution: Both medians are the same.

e. [2 points] A particle's position is given by the parametric equations $(x(t), y(t))$. If the graphs of $x(t)$ and $y(t)$ are given below, then the particle passes through the origin twice for $0 \le t \le 3$.

True $\boxed{\text{False}}$

Solution: The particle passes through the only at $t = 1$.





2. [7 points] For $n \geq 1$, consider the following sequences

- $a_n = (-1)^n + \frac{1}{n}$.
- $b_n = 1 + \frac{(-1)^n}{n}$.
- $c_n = (\frac{6}{5})^n$.
- $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$.

Circle your answers. No justification is needed.

1. Which sequences are bounded? a_n b_n c_n s_n None.
2. Which sequences are increasing? a_n b_n c_n s_n None.
3. Which sequences are convergent? a_n b_n c_n s_n None.

Solution:

1. a_n b_n s_n
2. c_n s_n
3. b_n s_n

3. [8 points] Let

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(x-2)^n$$

Find the interval of convergence of the power series. Justify your answer.

Solution: By the ratio test, the interval of convergence (except for the endpoints) consists of those x-values for which

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{4^{n+1}(n+1)^3}}{\frac{(x-2)^n}{4^n n^3}} \right| < 1$$

Simplifying, we get

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{4^{n+1}(n+1)^3}}{\frac{(x-2)^n}{4^n n^3}} \right| &< 1 \\
\lim_{n\to\infty} \left| \frac{(x-2)n^3}{4(n+1)^3} \right| &< 1 \\
|x-2| \lim_{n\to\infty} \left| \frac{n^3}{4(n^3+3n^2+3n+1)} \right| &< 1 \\
|x-2|\frac{1}{4} &< 1 \\
|x-2| &< 4
\end{aligned}$$

so the tentative interval of convergence is $-2 < x < 6$. We must check the endpoints

- **Endpoint $x = -2$:**

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(-2-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

converges by the alternating series test.

- **Endpoint $x = 6$:**

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(6-2)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by the p-test.

Therefore, the interval of convergence is $-2 \le x \le 6$.

4. [8 points] You are trapped on an island, and decide to build a signal fire to alert passing ships. You start the fire with 200 pounds of wood. During the course of a day, 40% of the wood pile burns away (so 60% remains). At the end of each day, you add another 200 pounds of wood to the pile. Let W_i denote the weight of the wood pile immediately after adding the i^{th} load of wood (the initial 200-pound pile counts as the first load).

a. [3 points] Find expressions for W_1, W_2 and W_3.

Solution:

$$W_1 = 200$$
$$W_2 = 200 + 200(0.6)$$
$$W_3 = 200 + 200(0.6) + 200(0.6)^2$$

b. [3 points] Find a closed form expression for W_n (*a closed form* expression means that your answer should not contain a large summation).

Solution:

$$W_n = \frac{200(1 - 0.6^n)}{1 - 0.6}$$

c. [2 points] Instead of starting with 200 pounds of wood and adding 200 pounds every day, you decide to start with P pounds of wood and add P pounds every day. If you plan to continue the fire indefinitely, determine the largest value of P for which the weight of the wood pile will never exceed 1000 pounds.

Solution:

$$\frac{P}{1 - 0.6} = 1000$$
$$P = 400$$

5. [14 points] Years later, after being rescued from the island, you design a machine that will automatically feed wood into a fire at a constant rate of 500 pounds per day. At the same time, as it burns, the weight of the wood pile (in pounds) decreases at a rate (in pounds/day) proportional to the current weight with constant of proportionality $\frac{1}{2}$.

a. [3 points] Let $W(t)$ be the weight of the wood pile t days after you start the machine. Write a differential equation satisfied by $W(t)$.

Solution:

$$W' = -0.5W + 500$$

b. [4 points] Find all equilibrium solutions to the differential equation in part (a). For each equilibrium solution, determine whether it is stable or unstable, and give a practical interpretation of its stability in terms of the weight of the wood pile as $t \to \infty$.

Solution: There is one equilibrium solution, $W = 1000$. This equilibrium solution is stable. This means that if your fire initially contains approximately 1000 pounds of wood, then in the long run the weight of the fire will approach 1000 pounds.

c. [7 points] Solve the differential equation from part (a), assuming that the wood pile weighs 200 pounds when you start the machine.

Solution:

$$\begin{aligned}
\frac{dW}{-0.5W + 500} &= dt \\
-2\ln|-0.5W + 500| &= t + C \\
-0.5W + 500 &= Ae^{-0.5t} \\
W &= 1000 + Ae^{-0.5t} \\
W &= 1000 - 800e^{-0.5t}
\end{aligned}$$

6. [9 points] The probability density function for the time t (in minutes) it takes for a pizza deliveryman to deliver a pizza is

$$p(t) = p_0\, t\, e^{-\lambda t^3}$$

This does *not* include the time required to prepare the pizza, only the time required to deliver it.

a. [4 points] Find a formula for the mean of $p(t)$ in terms of p_0 and λ. Your final answer should not contain any integrals.

Solution:

$$\begin{aligned} \text{mean} &= \int_0^\infty p_0 t^2 e^{-\lambda t^3}\, dt \\ &= \lim_{b\to\infty} \int_0^b p_0 t^2 e^{-\lambda t^3}\, dt \\ &= \lim_{b\to\infty} \left(\frac{-p_0}{3\lambda} e^{-\lambda t^3} \right) \Big|_0^b \\ &= \frac{p_0}{3\lambda} \end{aligned}$$

b. [3 points] The pizza parlor advertises that if you don't receive your pizza in 30 minutes or less after you order, then your pizza is free. After your order is taken, it takes 10 minutes for the chef to prepare a pizza for delivery. Write an expression for the probability that the next pizza that gets delivered will be free.

Solution:

$$1 - \int_0^{20} p_0 t e^{-\lambda t^3}\, dt \text{ or } \int_{20}^\infty p_0 t e^{-\lambda t^3}\, dt$$

c. [1 point] If $P(t)$ is the cumulative distribution of $p(t)$, find an expression for $P(20)$.

Solution:

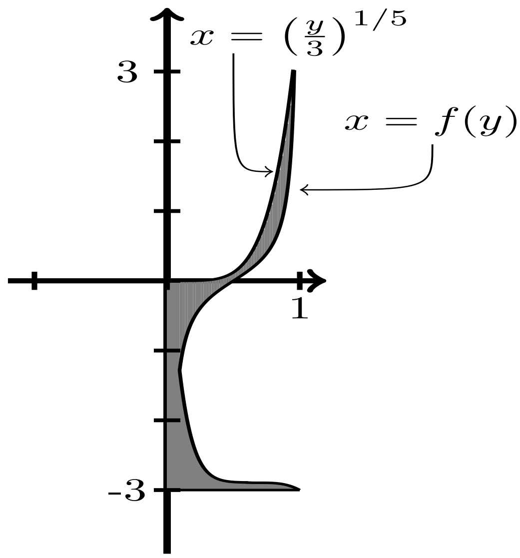
$$P(20) = \int_0^{20} p_0 t e^{-\lambda t^3}\, dt$$

d. [1 point] Explain the practical meaning of the number $P(20)$.

Solution: It is the probability that the next pizza gets delivered in less than 20 minutes. Alternatively, it is the probability that the next pizza *not* free.

7. [12 points]

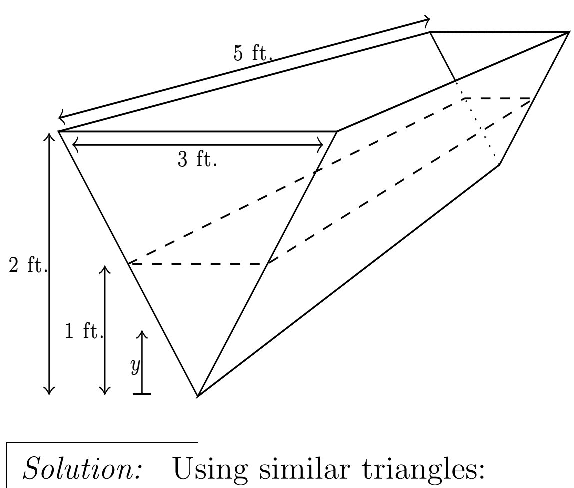
a. [5 points] You rotate the region shown about the y-axis to create a drinking glass. Write an expression that represents the volume of material required to construct the drinking glass (your answer may contain $f(y)$).



Solution:

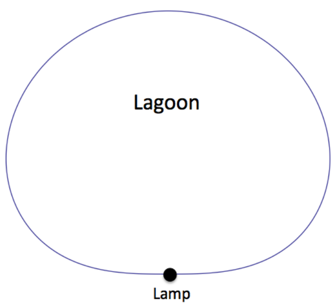
$$\text{Volume } = \int_0^3 \pi \left(f(y)^2 - \left(\frac{y}{3}\right)^{\frac{2}{5}} \right) dy + \int_{-3}^0 \pi f(y)^2 dy.$$

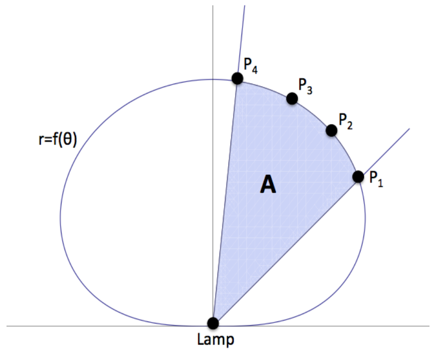
b. [7 points] Consider the vessel shown below. It is filled to a depth of 1 foot of water. Write an integral in terms of y (the distance in ft from the bottom of the vessel) for the work required to pump all the water to the top of the vessel. Water weighs 62.4 lbs/ft^3.



Solution: Using similar triangles:
Volume of a slice$=5\left(\frac{3}{2}y\right)\Delta y$
Work$=\int_0^1 5\left(\frac{3}{2}y\right)(62.4)(2-y)dy = \int_0^1 468y(2-y)dy.$

8. [13 points] A lamp is at the border of a small lagoon. The picture below shows the lamp at the origin and the lagoon described by the polar curve $r = f(\theta)$. During the night, the lamp illuminates the shaded region on the lagoon shown below.





	θ	$r = f(\theta)$
P_1	.8	.84
P_2	1	.91
P_3	1.2	.96
P_4	1.4	.99

a. [3 points] If A is the shaded area shown above, then $A = \int_a^b F(\theta)d\theta$ where

$F(\theta) =$________________ $a =$________ $b =$ ________.

Solution: $F(\theta) = \frac{1}{2}f(\theta)^2$ $a = .8$ $b = 1.4$

b. [5 points] Fill the table below with the values of $F(\theta)$. Then approximate the area of the shaded region using Left(3). Write all the terms in your sum.

θ	.8	1	1.2	1.4
$F(\theta)$				

Solution:

θ	.8	1	1.2	1.4
$F(\theta)$	.3528	.4141	.4608	.49

$$\text{Left}(3) = .2(.3528 + .4141 + .4608) = .2455.$$

c. [1 point] Is $F(\theta)$ increasing, decreasing or neither?

Solution: Increasing

d. [1 point] Is $F(\theta)$ concave up, concave down or neither?

Solution: Concave down

e. [3 points] Which Riemann sums (Left(3), Right(3) and/or Trap(3)) yield an under-estimate of the shaded area?

Solution: Left(3) and Trap(3).

9. [9 points]

a. [2 points] Find the Taylor series about $x = 0$ of $\sin(x^2)$. Your answer should include a formula for the general term in the series.

Solution:

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{4n+2}}{(2n+1)!} + \cdots$$

b. [2 points] Let m be a positive integer, find the Taylor series about $x = 0$ of $\cos(m\pi x)$. Your answer should include a formula for the general term in the series.

Solution:

$$\cos(m\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} = 1 - \frac{m^2\pi^2 x^2}{2!} + \cdots + \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} + \cdots$$

c. [5 points] Use the second degree Taylor polynomials of $\sin(x^2)$ and $\cos(m\pi x)$ to approximate the value of b_m, where

$$b_m = \int_{-1}^{1} \sin(x^2)\cos(m\pi x)dx.$$

(The number b_m is called a *Fourier coefficient of the function* $\sin x^2$. These numbers play a key role in *Fourier analysis*, a subject with widespread applications in engineering and the sciences.)

Solution:

$$b_m \approx \int_{-1}^{1} x^2 \left(1 - \frac{m^2\pi^2 x^2}{2!}\right) dx$$

$$b_m \approx \int_{-1}^{1} x^2 - \frac{m^2\pi^2}{2} x^4 dx$$

$$b_m \approx \frac{x^3}{3} - \frac{m^2\pi^2}{10} x^5 |_{-1}^{1}$$

$$b_m \approx \frac{2}{3} - \frac{m^2\pi^2}{5}$$

10. [10 points]

a. [5 points] Determine whether the following series converge or diverge (circle your answer). For each, justify your answer by writing what convergence rule or convergence test you would use to prove your answer. If you use the comparison test or limit comparison test, also write an appropriate comparison function.

1.[2 points]

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n^2}$$

Converge Diverge

Solution: **Diverges** $\lim_{n\to\infty}(-1)^n\frac{2^n}{n^2} \neq 0$.

2.[3 points]

$$\sum_{n=1}^{\infty}\frac{3n-2}{\sqrt{n^5+n^2}}$$

Converge Diverge

Solution: **Converges**
Limit Comparison Test $b_n = \frac{1}{n^{\frac{3}{2}}}$ (or a multiple) and p series $p = \frac{3}{2} > 1$.
Comparison Test $b_n = \frac{3}{n^{\frac{3}{2}}}$ and p series $p = \frac{3}{2} > 1$.

b. [5 points] Does the following series converge conditionally, absolutely, diverge or is it not possible to decide? Justify.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(1+\ln(n))}$$

Solution: Is it Absolute Convergent? Using Integral test with $f(x) = \frac{1}{x(1+\ln(x))}$

- $f(x) \geq 0$.
- $f(x)$ decreasing.

$$\sum_{n=1}^{\infty}\frac{1}{n(1+\ln(n))} \text{ behaves as } \int_1^{\infty}\frac{1}{x(1+\ln(x))}dx$$

$$\int_1^{\infty}\frac{1}{x(1+\ln(x))}dx = \lim_{b\to\infty}\int_1^{b}\frac{1}{x(1+\ln(x))}dx = \lim_{b\to\infty}\ln|1+\ln(b)| = \infty$$

Hence $\sum_{n=1}^{\infty}\frac{1}{n(1+\ln(n))}$ is not absolutely convergent.
Is it Conditionally Convergent? Since $a_n = \frac{1}{n(1+\ln(n))}$ decreasing and converges to zero, then by Alternating series test
$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(1+\ln(n))}$ converges conditionally.