Multiple Choice

(5 points each)

1. What is the power series representation of $f(x) = \ln(1+x)$ at $x = 0$?

(a) $\sum_{n=0}^{\infty} x^n$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$

(d) $\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^n$ (e) $\sum_{n=1}^{\infty} \ln(1+n) x^n$

$$f'(x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} + C$$

$f(0) = C$, but $f(0) = \ln 1 = 0$, so $C = 0$.

2. Find the series which converges but does not converge absolutely.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ (b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (c) $\sum_{k=2}^{\infty} (-1)^k \left(1 - \frac{1}{k}\right)$

(d) $\sum_{k=0}^{\infty} (-1)^k e^{-k}$ (e) $\sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)^k$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges by the Alternating Series Test.

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by the p-series test.

3. Which series converges absolutely?

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ (b) $\sum_{k=1}^{\infty} k(-1)^{k+1}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

(d) $\sum_{k=1}^{\infty} \left(\sqrt{k+1} - \sqrt{k}\right)$ (e) $\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$ converges absolutely because $\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1/3}{1-1/3} = \frac{1}{2}$.

4. Find the vector projection of $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ onto $2\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$.

(a) -7 (b) $-\frac{7}{38}(2\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})$ (c) $-\frac{7}{\sqrt{38}}$ (d) $-\frac{1}{2}(3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ (e) $-\frac{7}{\sqrt{14}}$

The vector projection is

$$\frac{(3\vec{i} + 2\vec{j} - \vec{k}) \cdot (2\vec{i} - 5\vec{j} + 3\vec{k})}{\| 2\vec{i} - 5\vec{j} + 3\vec{k} \|^2} (2\vec{i} - 5\vec{j} + 3\vec{k}) = \frac{-7}{38}(2\vec{i} - 5\vec{j} + 3\vec{k}).$$

5. Find the unit vector in the direction of $3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k}$.

(a) $\frac{1}{169}(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$ (b) $\frac{1}{169}(-4\boldsymbol{i} + 12\boldsymbol{j} + 3\boldsymbol{k})$ (c) $13(-12\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

(d) $\frac{1}{13}(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$ (e) $\frac{1}{5}(-3\boldsymbol{i} - 4\boldsymbol{j} + 12\boldsymbol{k})$

The unit vector is $\dfrac{1}{\|3\vec{i}+4\vec{j}-12\vec{k}\|}(3\vec{i}+4\vec{j}-12\vec{k}) = \frac{1}{13}(3\vec{i}+4\vec{j}-12\vec{k})$.

6. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n}(2x-1)^n$.

(a) 3 (b) $\frac{1}{2}$ (c) 6 (d) $\frac{1}{3}$ (e) ∞

$\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n}(2x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}(x-\tfrac{1}{2})^n$, so $c_n = \frac{(-1)^n}{3^n}$.

$$r = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{1/3^n}{1/3^{n+1}} = \lim_{n\to\infty} 3 = 3.$$

7. Find the equation of the sphere passing through (7,3,5) with center $(9, 0, -1)$.

(a) $(x-7)^2 + (y-3)^2 + (z-5)^2 = 82$
(b) $x + y + z = 15$
(c) $7(x-9) + 3y + 5(z+1) = 0$
(d) $(x-9)^2 + y^2 + (z+1)^2 = 83$
(e) $(x-9)^2 + y^2 + (z+1)^2 = 49$

The radius is equal to dist$((7,3,5),(9,0,-1)) = 7$, so the equation is just

$$(x-9)^2 + (y-0)^2 + (z-(-1))^2 = 7^2$$

8. Consider the 4th partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$ as an approximation. Use the alternating series rule to obtain an upper bound on the absolute value of the error.

(a) $\frac{1}{64}$ (b) $\int_4^{\infty} \frac{dx}{x2^x}$ (c) $\frac{1}{384}$ (d) $\sum_{n=5}^{\infty} \frac{(-1)^n}{n2^n}$ (e) $\frac{1}{160}$

$$|\text{Error}| \leq |\text{5th term}| = \frac{1}{5 \cdot 2^5} = \frac{1}{160}$$

9. For which series is the ratio test inconclusive?

(a) $\sum_{k=1}^{\infty} \frac{k^3}{3^k}$ (b) $\sum_{k=1}^{\infty} \frac{2^k}{k^2}$ (c) $\sum_{k=1}^{\infty} \frac{3^{2k+1}}{2^{3k+5}}$ (d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ (e) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{10}}$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{1/(k+1)^{10}}{1/k^{10}} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^{10} = 1$$

10. If the force $\boldsymbol{F} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ is applied to a body that is constrained to move along the line segment from $A(1,-2,0)$ to $B(3,-5,2)$, which number is the cosine of the angle between $\boldsymbol{F}$ and this directed line segment?

(a) $\frac{13}{14}$ (b) $\frac{14}{\sqrt{13}\sqrt{17}}$ (c) $\frac{13}{\sqrt{14}\sqrt{17}}$ (d) $\frac{17}{\sqrt{13}\sqrt{14}}$ (e) $\frac{13}{17}$

$$\cos\theta = \frac{\vec{F}\cdot\overrightarrow{AB}}{\|\vec{F}\|\,\|\overrightarrow{AB}\|} = \frac{(3\vec{i}-\vec{j}+2\vec{k})\cdot(2\vec{i}-3\vec{j}+2\vec{k})}{\|3\vec{i}-\vec{j}+2\vec{k}\|\;\|2\vec{i}-3\vec{j}+2\vec{k}\|} = \frac{13}{\sqrt{14}\sqrt{17}}$$

Work-Out Problems

Show your work. No credit will be given to an unsupported answer. Partial credit is possible.

11. (5 points each) Find the Maclaurin series for each function.

(a) $f(x) = x^2 e^{2x}$

$$e^{\mu} = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \Rightarrow e^{2x} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$$

$$\Rightarrow x^2 e^{2x} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^{k+2}$$

(b) $f(x) = \dfrac{\sin x}{x}$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \Rightarrow \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

(c) $f(x) = \dfrac{1}{1+4x^2}$

$$\frac{1}{1-\mu} = \sum_{k=0}^{\infty} \mu^k \Rightarrow \frac{1}{1+4x^2} = \sum_{k=0}^{\infty} (-4x^2)^k = \sum_{k=0}^{\infty} (-4)^k x^{2k}$$

12. (10 points) Find the interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{1}{n3^{2n}}(x-1)^n.$$

$$r = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{1/n3^{2n}}{1/(n+1)3^{2(n+1)}}$$

$$= \lim_{n\to\infty} \frac{(n+1)3^{2n+2}}{n3^{2n}}$$

$$= 3^2 \lim_{n\to\infty} \frac{n+1}{n} = 9$$

We have absolute convergence for $|x-1|<9$ — i.e., for

$$-8 < x < 10.$$

Check the endpoint $x=10$. The series reduces to $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by the p-series test, so the interval of convergence does not include 10.

Check the endpoint $x=-8$. The series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Since the sequence $\left(\frac{1}{n}\right)$ is decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$, this series converges by the Alternating Series Test.

Therefore, the interval of convergence is $[-8, 10)$ — i.e., the range $-8 \leq x < 10$.

13. Consider the function $f(x) = \sqrt{x}$.

(a) (6 points) Find the 2nd degree Taylor polynomial $T_2(x)$ at $x = \frac{9}{4}$.

$f(x) = x^{1/2}$ $\quad f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f''(x) = -\frac{1}{4}x^{-3/2}$

$f\left(\frac{9}{4}\right) = \frac{3}{2}$ $\quad f'\left(\frac{9}{4}\right) = \frac{1}{3}$ $\quad f''\left(\frac{9}{4}\right) = -\frac{2}{27}$

$$T_2(x) = f\left(\tfrac{9}{4}\right) + f'\left(\tfrac{9}{4}\right)\left(x-\tfrac{9}{4}\right) + \tfrac{1}{2}f''\left(\tfrac{9}{4}\right)\left(x-\tfrac{9}{4}\right)^2$$
$$= \frac{3}{2} + \frac{1}{3}\left(x-\frac{9}{4}\right) - \frac{1}{27}\left(x-\frac{9}{4}\right)^2$$

(b) (6 points) Consider the interval $1 \le x \le 3$ (which contains $\frac{9}{4}$) together with the Taylor inequality

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}\left|x - \frac{9}{4}\right|^{n+1}$$

over this interval, where

$$M = \max_{1\le x\le 3} |f^{(n+1)}(x)|.$$

For $f(x) = \sqrt{x}$, use this inequality to find a number κ such that

$$|f(x) - T_2(x)| \le \kappa, \qquad 1 \le x \le 3.$$

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

$$\max_{1\le x\le 3} |f'''(x)| = \frac{3}{8}\max_{1\le x\le 3} x^{-5/2} = \frac{3}{8}$$

$$|f(x) - T_2(x)| \le \frac{3/8}{3!}\left|x-\frac{9}{4}\right|^3 = \frac{1}{16}\left|x-\frac{9}{4}\right|^3$$

For $1 \le x \le 3$, $\left|x - \frac{9}{4}\right| \le 5/4$

$$|f(x) - T_2(x)| \le \frac{1}{16}\left(\frac{5}{4}\right)^3 = \frac{125}{1024}$$

14. (6 points) Calculate the 3rd-degree Taylor polynomial of

$$f(x) = x^5 - 4x^4 + x^2 - 8x + 5$$

at $x = 1$.

$f'(x) = 5x^4 - 16x^3 + 2x - 8$

$f''(x) = 20x^3 - 48x^2 + 2$

$f'''(x) = 60x^2 - 96x$

$f(1) = -5 \quad f'(1) = -17 \quad f''(1) = -26 \quad f'''(1) = -36$

$T_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3$

$T_3(x) = -5 - 17(x-1) - 13(x-1)^2 - 6(x-1)^3$

15. (7 points) Express $\int_0^b e^{-x^2}\, dx$ as an infinite series by using the Maclaurin series of the integrand.

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} \Rightarrow e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}$$

$$\Rightarrow \int_0^b e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^b x^{2k} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k+1}}{k!(2k+1)}$$