1) The following differential equation is a model for logistic growth of a population P.

$$\frac{dP}{dt} = (100 - P)P$$

a) Determine the equilibrium values (also called rest points) for this model.

$\frac{dP}{dt} = 0$ when $P = 0$ and when $P = 100$

b) Determine the value of P when the population is increasing most rapidly.

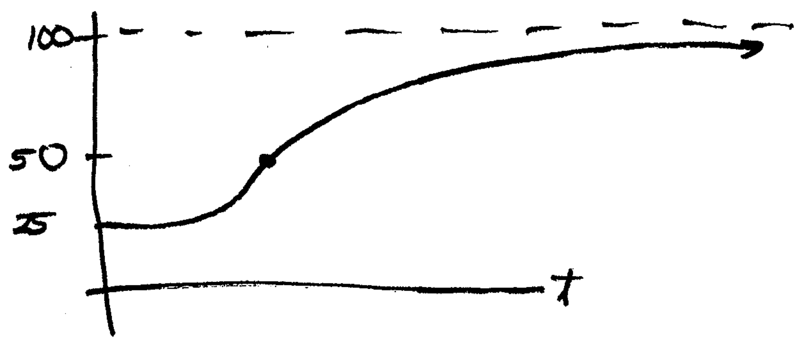
When is $\frac{dP}{dt}$ maximum?

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left[(100-P)P\right] = \left[(100-P)\frac{dP}{dt} + P\left(-\frac{dP}{dt}\right)\right]$$

$$= 100\frac{dP}{dt} - 2P\frac{dP}{dt} = (100-2P)\frac{dP}{dt}$$

Max will be when $100 - 2P = 0$ $\boxed{P = 50.}$ Min is at $\frac{dP}{dt} = 0$ or $P = 0, 100$

c) Sketch the solution curve if the initial population is $P_0 = 25$.



Note: The solution of the separable DE is

$$P(t) = \frac{100}{1 + 3e^{-100t}} \quad \text{Note } P(0) = 25$$

$P = 50$ occurs at $t = \frac{\ln 3}{100}$

2)(a) Find the work done by a force $F(x) = \dfrac{2x-5}{(x-3)(x-2)}$ in moving an object from $x = 4$ to $x = 6$.

$$W[4,6] = \int_4^6 \frac{2x-5}{(x-3)(x-2)}\,dx = \int_4^6 \frac{1}{x-3} + \frac{1}{x-2}\,dx$$
$$= \ln(x-3) + \ln(x-2) \Big|_4^6$$
$$= \ln 3 - \ln 1 + \ln 4 - \ln 2 \qquad (\ln 1 = 0)$$
$$= \ln\left(\frac{3 \cdot 4}{2}\right) = \ln 6$$

(b) During the decade of the 1980's, the number of AIDS cases in the United States grew continuously at a rate proportional to the number of current cases. In 1980, there were 200 reported AIDS victims. That number grew to 2000 by 1983. According to this model, how many cases of AIDS were there in the U.S. in 1989?

Let $y(t)$ = # Aids cases at time t years (use 1980 as baseline)
where 1980 is $t=0$, 1983 is $t=3$, 1989 is $t=9$.

exponential growth model

$y(0) = 200$
$y(3) = 2000$
$y(9) = ?$

$\frac{dy}{dt} = ky$ has solution $y(t) = y_0 e^{kt}$

$y_0 = 200$

For 1983, $2000 = 200 e^{k \cdot 3}$
$e^{3k} = 10$
$3k = \ln 10$
$k = \frac{\ln 10}{3}$

$$\boxed{y(t) = 200 e^{\frac{\ln 10}{3} t}}$$

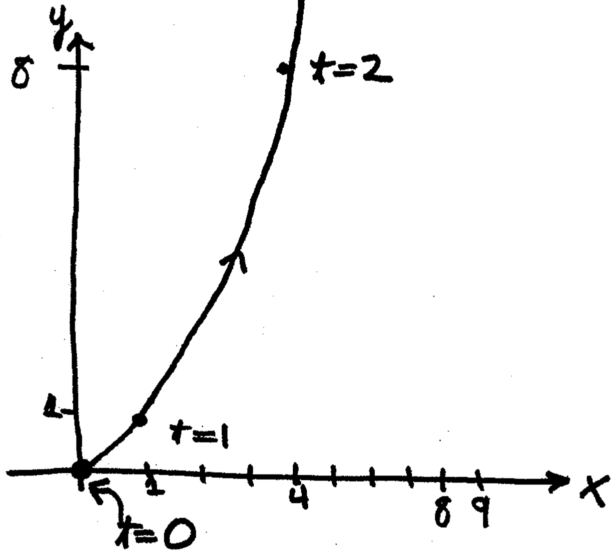
$$y(9) = 200 e^{\frac{\ln 10}{3} \cdot 9} = 200 e^{\ln(10^3)}$$
$$= 200 \times 1000 = \boxed{200{,}000}$$

Note under the model, # cases is multiplied by 10 every 3 years

$(x,y)=(9,27)$ at $t=3$

3) A path of an object moving in the x-y plane has parametric form $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$ for $0 \leq t \leq 3$.

(a) Using the parameter t = *time* (in seconds), sketch a graph in the x-y plane of the path of motion of the particle and show the direction of motion.



b) Find the length of the path followed during the 3-second interval.

$$L[0,3] = \int_{t=0}^{t=3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^3 \sqrt{(2t)^2 + (3t^2)^2}\, dt.$$

$$= \int_0^3 \sqrt{4t^2 + 9t^4}\, dt.$$

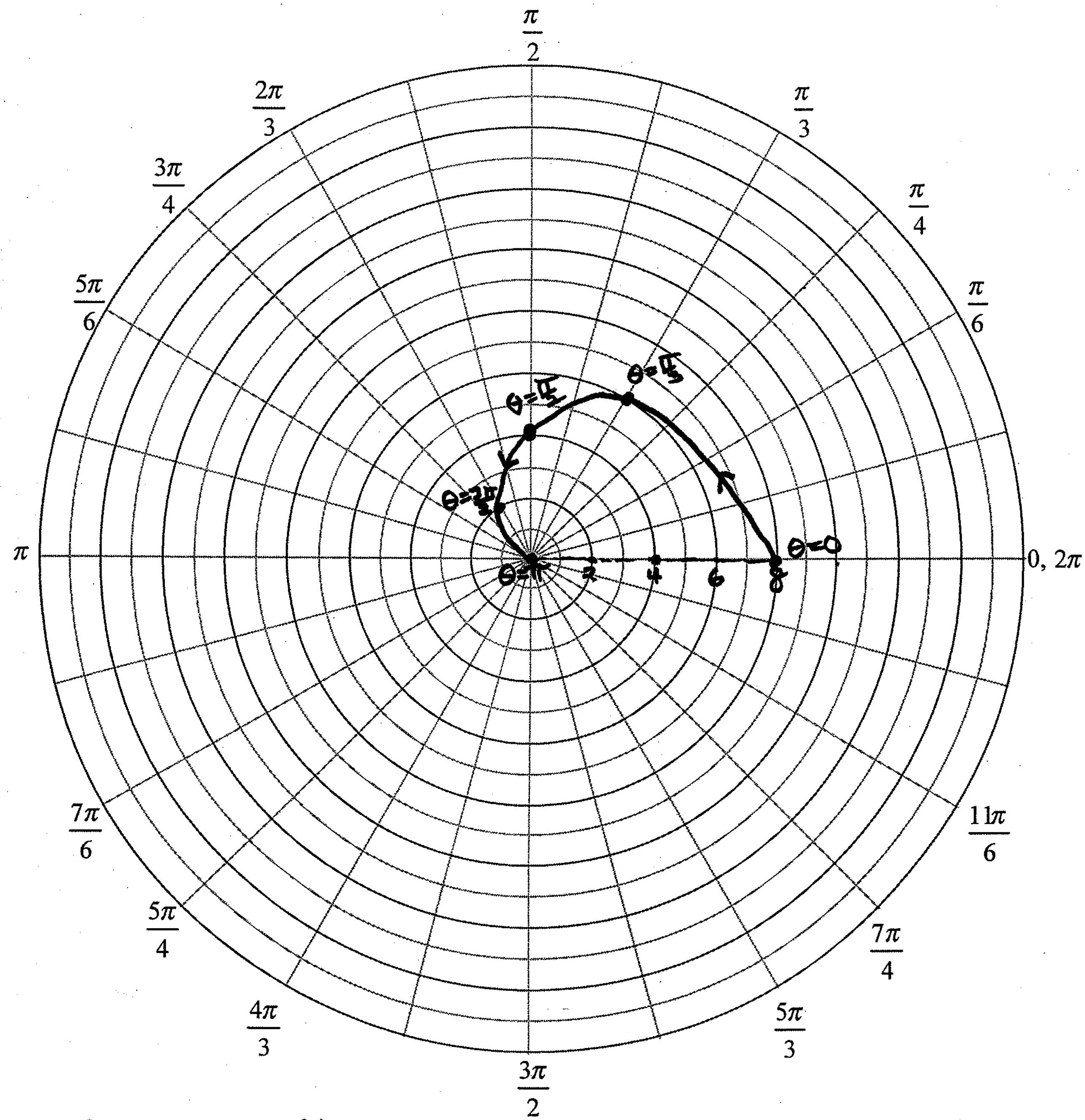
$$= \int_0^3 2t\sqrt{1 + \tfrac{9}{4}t^2}\, dt \quad \text{let } u = \tfrac{9}{4}t^2 \quad du = \tfrac{9}{2}t\, dt$$

Second solution: $t = \sqrt{x}$ and so $y = t^3 = x^{\frac{3}{2}}$. $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$$L\overset{x:}{[0,9]} = \int_{x=0}^{x=9} \sqrt{1 + \left(\tfrac{3}{2}\sqrt{x}\right)^2}\, dx = \int_0^9 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

$$= \frac{4}{9}\, \frac{\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^9 = \frac{8}{27}\left[\left(1+\tfrac{81}{4}\right)^{\frac{3}{2}} - 1\right] = \frac{8}{27}\left[\left(\tfrac{85}{4}\right)^{\frac{3}{2}} - 1\right]$$

4) Find the area of the region in the upper half plane enclosed by the cardioid $r = 4(1 + \cos(\theta))$. Give a sketch of this area on the attached polar graph paper.



$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta = \int_0^\pi \frac{4^2}{2}(1+\cos\theta)^2 \, d\theta = 8\int_0^\pi 1 + 2\cos\theta + \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} \, d\theta$$

$$= 8\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^\pi$$

$$= 12\pi$$

5) Evaluate each of the following:

(a) dy/dx if $y = \ln(\sec x)$

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

Needs $\sec x > 0$ or $\cos x > 0$ for example $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $f'(x)$ if $f(x) = e^{x^2} \arctan x$ (also written $f(x) = e^{x^2} \tan^{-1} x$)

$$f'(x) = e^{x^2} \frac{1}{1+x^2} + \tan^{-1} x \cdot e^{x^2} \cdot 2x$$ by product rule and chain rule.

$$= e^{x^2}\left[\frac{1}{1+x^2} + 2x \tan^{-1} x\right]$$

(c) $\lim_{x\to\infty}\left((\cos(1/x))^x\right)$ behavior is "1^∞", an indeterminate form

Let $y = (\cos(\frac{1}{x}))^x$ and consider $\ln(y) = x \ln(\cos \frac{1}{x})$

$$= \frac{\ln(\cos \frac{1}{x})}{\frac{1}{x}}$$

Now $\lim_{x\to\infty} \ln(y) = $ "$\frac{0}{0}$" so we can use l'Hôpital.

$$\hookrightarrow = \lim_{x\to\infty} \frac{\frac{1}{\cos(\frac{1}{x})} \cdot -\sin(\frac{1}{x})\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\frac{-1}{x^2}}} = \frac{1}{1} \cdot -0 = 0$$

Thus $\lim_{x\to\infty} y = e^0 = 1.$

(d) $\sum_{k=4}^{\infty}\left(\frac{-1}{\pi}\right)^k$ $= \frac{1}{\pi^4} - \frac{1}{\pi^5} + \frac{1}{\pi^6} - \frac{1}{\pi^7} + \cdots$ geometric with $a = \frac{1}{\pi^4}$ and $r = -\frac{1}{\pi}$

$$s = \frac{a}{1-r} = \frac{\frac{1}{\pi^4}}{1 - -\frac{1}{\pi}} = \frac{1}{\pi^4} \cdot \frac{\pi}{\pi+1} = \frac{1}{\pi^3(1+\pi)}$$

6) Integrate

(a) $\int 3x\sin(x^2)dx$ $u=x^2$ $du=2xdx$ so $xdx=\frac{1}{2}du$.

u-sub $\int \frac{3}{2}\sin(u)du = -\frac{3}{2}\cos(u)+C$ C any constant.

$= -\frac{3}{2}\cos(x^2)+C$

(b) $\int x^2 \sin x\, dx$ Use parts 2 times: $u=x^2$ $dv=\sin x\,dx$

$du=2x\,dx$ $v=-\cos x$.

$\hookrightarrow = -x^2\cos x - \int -2x\cos x\, dx$. Now for second integral:

$u=x$ $dv=\cos x\,dx$

$du=dx$ $v=\sin x$.

$= -x^2\cos x + 2\left[x\sin x - \int \sin x\, dx\right]$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$.

(c) $\int \sin^2 5x\, dx$ $= \int \frac{1-\cos(10x)}{2}dx = \frac{1}{2}x - \frac{1}{2}\frac{\sin(10x)}{10} + C$

$= \frac{x}{2} - \frac{1}{20}\sin(10x)+C$

(d) This problem has no elementary solution. Express your answer as a series:

$\int \sin(x^2)\, dx$ Use Taylor Series $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

$\int \sin(x^2)dx = \frac{x^3}{3} - \frac{x^7}{3!\cdot 7} + \frac{x^{11}}{5!\,11} - \frac{x^{15}}{7!\,15} + \cdots + C$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} + C$$

7) Mark True or False. If the statement is false indicate how the statement could be changed to provide a true statement, or provide an example that demonstrates the falsity of the statement.

F (a) Every bounded sequence converges.

0, 1, 0, 1, 0, 1, ... is bounded above by 1, yet diverges.

T (b) Every convergent sequence is bounded.

All but first finitely many terms are between $L-1$ and $L+1$ where L is limit.

T (c) An infinite series converges if and only its sequence of partial sums converges.

Definition of convergence of a series.

F (d) An infinite series $\sum a_k$ converges if and only if $\lim_{k\to\infty} a_k = 0$

$\Rightarrow$ True

$\Leftarrow$ False, harmonic series is counterexample.

F (e) If a series of positive terms $\sum a_k$ converges, then $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$

Ratio test cannot always show convergence.

Example: $a_n = \frac{1}{n^2}$ converges but ratio test gives limit of 1

F (f) An alternating series $\sum (-1)^k a_k$ where each $a_k > 0$ converges if the terms a_k are strictly decreasing.

Terms must also approach 0.

Suppose $a_n = 1 + \frac{1}{n}$. Terms decrease yet series diverges.

T (g) Every absolutely convergent series converges.

Theorem (used very often!)

T (h) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all real values of x.

most useful Taylor series.

T (i) $\ln x^2 = \int_1^x \frac{2}{t}\,dt$ for all $x > 0$.

$\ln x^2 = 2\ln x = 2\int_1^x \frac{1}{t}\,dt$ by def. of $\ln(x)$

T (j) If $F(x) = \int_0^{7x} \sin(t^2)\,dt$, the derivative is $F'(x) = 7\sin(49x^2)$.

$\frac{d}{dx}\int_0^{7x} \sin(t^2)\,dt = \sin((7x)^2)\cdot\frac{d}{dx}(7x)$ by FTofC and Chain Rule

$= 7\cdot\sin(49x^2)$

8) Determine the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{(x-4)^k}{3^k(k+3)}$

Be sure to discuss any endpoints that may appear.

Use absolute ratio test

$$\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\frac{|x-4|^{k+1}}{3^{k+1}\,k+4}}{\frac{|x-4|^k}{3^k\,k+3}} = \lim_{k\to\infty} \frac{|x-4|}{3} \cdot \frac{k+3}{k+4}$$

$$= \frac{|x-4|}{3}$$

Now $\frac{|x-4|}{3} < 1$ when $|x-4| < 3$ or $-3 < x-4 < 3$

$1 < x < 7$

Test endpoints 1 and 7.

$x = 7$ gives $\sum_{k=0}^{\infty} \frac{(7-4)^k}{3^k(k+3)} = \sum_{k=0} \frac{1}{k+3}$ diverges as "tail" of the harmonic series.

$x = 1$ $\sum_{k=0}^{\infty} \frac{(1-4)^k}{3^k(k+3)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+3} = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

converges as alternating harmonic series

$I = [1, 7)$ is interval of convergence.

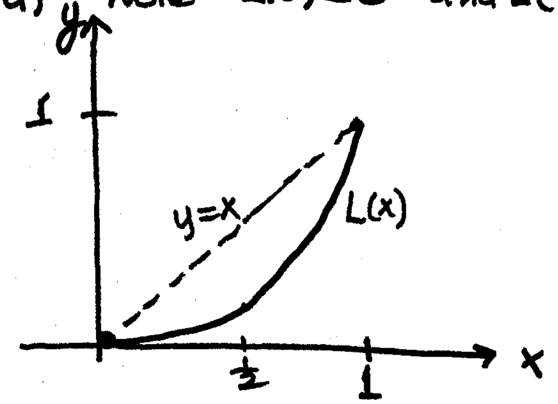
Have absolute convergence except at $x = 1$.

9) An economist suggested that the distribution of individual income in the United States in 2005 was well described by the Lorenz curve

$$L(x) = \frac{2x^3}{1+x^4}.$$

(a) Plot L(x).
(b) Determine how much of the wealth is owned by the richest one-half of the population.
(c) Find the Gini index, C, for this Lorenz function.

a) Note $L(0)=0$ and $L(1)=1$ and $L(\frac{1}{2})=\dfrac{\frac{2}{8}}{1+\frac{1}{16}}=\frac{4}{17}$



b) $L(\frac{1}{2})=\frac{4}{17}$ so poorest half of population has $\frac{4}{17}$ of wealth.
Richest half of population has $\frac{13}{17}$ of wealth (76.5%)

c) $C=\dfrac{\int_0^1 x-\frac{2x^3}{1+x^4}\,dx}{\frac{1}{2}}$ $u=1+x^4$, $du=4x^3dx$.

$$=2\left[\frac{x^2}{2}-\frac{1}{2}\ln(1+x^4)\right]_0^1$$

$$=2\left[\frac{1}{2}-\frac{1}{2}\ln 2-0\right]$$

$$=1-\ln 2 \doteq .307$$

larger C means greater inequality.
.307 is a "typical" value for nations.