1. (16 pts) Do the following sequences converge or diverge? If convergent, give the limit. Explicitly show your reasoning.

(a) $a_n = (-1)^n \dfrac{n}{n+1}$ (b) $\left\{(-1)^n \dfrac{n}{n^2+1}\right\}_{n\geq 22}$

Solution:

(a)Sequence diverges since $\dfrac{n}{n+1} \to 1$ as $n \to \infty$ so $a_n = (-1)^n \dfrac{n}{n+1}$ oscillates between -1 and 1.

(b)Sequence converges to 0 by Squeeze Theorem since,

$$\frac{-n}{n^2+1} \leq \frac{(-1)^n n}{n^2+1} \leq \frac{n}{n^2+1} \text{ for } n \geq 1$$

and $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{-n}{n^2+1} \underbrace{=}_{L'H} 0.$

---

2. (20 pts) For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=10}^{\infty} \frac{\cos(n\pi)}{n^{3/2}-1}$ (b) $\displaystyle\sum_{n=21}^{\infty} \frac{(-1)^n}{\ln(n)+n}$ (c) $\displaystyle\sum_{n=88}^{\infty} ne^{-n}$

Solution:

(a) Series is <u>absolutely convergent</u> since

$$\sum_{n=10}^{\infty} \left|\frac{\cos(n\pi)}{n^{3/2}-1}\right| = \sum_{n=10}^{\infty} \left|\frac{(-1)^n}{n^{3/2}-1}\right| = \sum_{n=10}^{\infty} \frac{1}{n^{3/2}-1}$$

and $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^{3/2}-1}$ converges by Limit Comparison Test with $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series.

(b) Series is <u>conditionally convergent</u> since for $n \geq 1$ we have $\ln(n) < n$ implies $\ln(n) + n < 2n$, and

$$\sum_{n=21}^{\infty} \left|\frac{(-1)^n}{\ln(n)+n}\right| = \sum_{n=21}^{\infty} \frac{1}{\ln(n)+n} > \sum_{n=21}^{\infty} \frac{1}{2n}$$

so $\displaystyle\sum_{n=21}^{\infty} \frac{1}{\ln(n)+n}$ diverges by Direct Comparison with $\displaystyle\sum_{n=21}^{\infty} \frac{1}{2n}$ which is divergent (since it is a nonzero multiple of a divergent harmonic series.) The alternating series $\displaystyle\sum_{n=21}^{\infty} \frac{(-1)^n}{\ln(n)+n}$ converges by the Alternating Series Test since $u_n = \dfrac{1}{\ln(n)+n}$ and note that $u_{n+1} = \dfrac{1}{\ln(n+1)+(n+1)} < \dfrac{1}{\ln(n)+n} = u_n$ and $\lim\limits_{n\to\infty} u_n = 0$.

(c) <u>Absolutely convergent</u> by Ratio (or Root) test since,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{e} = \frac{1}{e} < 1$$

---

3. (20 pts) Show all work.

   (a) Use series to evaluate: $\lim_{x\to 0} \frac{\sin(x) - \tan^{-1}(x)}{x^3}$.

   (b) Find the **Taylor Polynomial of order 2** generated by $f(x) = (1-x)^{1/2}$ at $x = 0$.

Solution:

(a) Converges to 1/6 since,

$\lim_{x\to 0} \frac{\sin(x)-\tan^{-1}(x)}{x^3} = \lim_{x\to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right)}{x^3}$, and moreover

$\lim_{x\to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right)}{x^3} = \lim_{x\to 0} \frac{\left(\frac{x^3}{3}-\frac{x^3}{3!}\right)+\left(\frac{x^5}{5!}-\frac{x^5}{5}\right)+\left(\frac{x^7}{7}-\frac{x^7}{7!}\right)+\cdots}{x^3}$, and finally

$\lim_{x\to 0} \frac{\left(\frac{x^3}{3}-\frac{x^3}{3!}\right)+\left(\frac{x^5}{5!}-\frac{x^5}{5}\right)+\left(\frac{x^7}{7}-\frac{x^7}{7!}\right)+\cdots}{x^3} = \lim_{x\to 0} \frac{x^3\left(\left(\frac{1}{3}-\frac{1}{3!}\right)+\left(\frac{x^2}{5!}-\frac{x^2}{5}\right)+\left(\frac{x^4}{7}-\frac{x^4}{7!}\right)+\cdots\right)}{x^3} = \frac{1}{3} - \frac{1}{3!} = \frac{1}{6}$

(b) Note,

$$(1-x)^{1/2} = 1 + \sum_{k=1}^{\infty} \binom{1/2}{k}(-x)^k = 1 - \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \cdots = 1 - \frac{x}{2} + \frac{(1/2)(-1/2)}{2!}x^2 + \cdots$$

so $P_2(x) = 1 - \frac{x}{2} - \frac{x^2}{8}$.

---

4. (24 pts) Justify your answers.

   (a) Find a Maclaurin Series for $f(x) = xe^{-x/2}$. (*You may use your knowledge of the Maclaurin Series of* e^x *to answer this question.*)

   (b) Use the first 3 non-zero terms of the series found in part (a) to approximate $\int_0^1 xe^{-x/2}\,dx$.

   (c) Estimate the error of the approximation found in part (b).

   (d) Is the approximation found in part (b) an *underestimate* or an *overestimate*?

Solution:

(a) $xe^{-x/2} = x \cdot \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^n n!} = x - \frac{x^2}{2} + \frac{x^3}{2^2 2!} - \frac{x^4}{2^3 3!} + \cdots$

(b) $\int_0^1 xe^{-x/2}\,dx \approx \int_0^1 \left(x - \frac{x^2}{2} + \frac{x^3}{8}\right)dx = \left(\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{32}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{6} + \frac{1}{32}$

(c) $\int_0^1 xe^{-x/2}\,dx = \frac{1}{2} - \frac{1}{6} + \frac{1}{32} - \frac{1}{5\cdot 2^3 \cdot 3!} + \cdots$, so $|\text{Error}| \le \left| -\frac{1}{5\cdot 2^3\cdot 3!}\right| = \frac{1}{240}$
(One could also use the Taylor Remainder Term $R_3(x)$ to estimate the error.)

(d) Note, $-\frac{1}{5\cdot 2^3\cdot 3!} < 0$ implies the approximation used in part (b) is an overestimate.

---

5. (20 pts) Show all work.

   (a) Find the interval of convergence of $\sum_{n=0}^{\infty}(n+1)x^n$.

   (b) For what values of x is the series conditionally convergent? absolutely convergent? divergent?

(c) Show that the series in part (a) converges to $\frac{1}{(1-x)^2}$.

(d) Find the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{3^n}$.

Solution:

(a)Using Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x|\cdot\lim_{n\to\infty}\frac{n+2}{n+1} = |x|$$

so letting $|x| < 1$ implies $-1 < x < 1$ and when $x = 1$ we have $\sum_{n=0}^{\infty}(n+1)$ which diverges by the n$^{\text{th}}$ Term Divergence Test, and if $x = -1$ we have $\sum_{n=0}^{\infty}(-1)^n(n+1)$ which also diverges by the n$^{\text{th}}$ Term Divergence Test, so the Interval of Convergence is $(-1, 1)$.

(b) (i) Not conditionally convergent for any values of x.

(ii) Absolutely convergent for $-1 < x < 1$.

(iii) Divergent on $(-\infty, -1] \cup [1, \infty)$

(c) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$

(One could also use a Binomial Series expansion here.)

(d) Series converges to $9/4$ since

$$\sum_{n=0}^{\infty}\frac{n+1}{3^n} = \sum_{n=0}^{\infty}(n+1)x^n\bigg|_{x=1/3} = \frac{1}{(1-x)^2}\bigg|_{x=1/3} = \frac{1}{(1-1/3)^2} = 9/4.$$