Solutions to Final Exam

1. Evaluate $\int \sin(\ln t)\,dt$.

Solution. We first make the substitution $t = e^x$, for which $dt = e^x\,dx$. This gives

$$\int \sin(\ln t)\,dt = \int e^x \sin(x)\,dx.$$

To evaluate the integral on the right we integrate by parts twice. First we use

$$u = e^x, \qquad dv = \sin(x)\,dx$$
$$du = e^x\,dx \qquad v = -\cos(x).$$

This gives

$$\int e^x \sin(x) = -e^x \cos(x) + \int e^x \cos(x)\,dx.$$

We now use

$$u = e^x, \qquad dv = \cos(x)\,dx$$
$$du = e^x\,dx \qquad v = \sin(x).$$

This gives

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx.$$

Moving the integral on the right to the other side of the equation and dividing by 2 gives

$$\int e^x \sin(x)\,dx = \frac{e^x \sin(x) - e^x \cos(x)}{2}.$$

Remember, we now have to add on a $+C$ since this is an indefinite integral. Going back to the original integral, we now have

$$\int \sin(\ln t)\,dt = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C$$

where $t = e^x$. Rewriting in terms of t gives the final answer:

$$\boxed{\int \sin(\ln t)\,dt = \frac{t\sin(\ln t) - t\cos(\ln t)}{2} + C.}$$

2. Evaluate $\displaystyle\int \frac{dx}{\sqrt{9x^2+16}}$.

Solution. We first make the substitution $x = \frac{4}{3}\tan(\theta)$. Note that we then have

$$dx = \tfrac{4}{3}\sec^2(\theta)\,d\theta \quad \text{and} \quad 9x^2+16 = 16\sec^2(\theta).$$

We therefore have

$$\int \frac{dx}{\sqrt{9x^2+16}} = \int \frac{\frac{4}{3}\sec^2(\theta)}{\sqrt{16\sec^2(\theta)}}\,d\theta = \frac{1}{3}\int \sec(\theta)\,d\theta = \tfrac{1}{3}\ln(\sec\theta + \tan\theta) + C.$$

We must now rewrite the answer in terms of x. We have

$$\tan(\theta) = \tfrac{3}{4}x \quad \text{and} \quad \sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+\tfrac{9}{16}x^2}.$$

Therefore we have the final answer

$$\boxed{\int \frac{dx}{\sqrt{9x^2+16}} = \tfrac{1}{3}\ln\left(\tfrac{3}{4}x + \sqrt{1+\tfrac{9}{16}x^2}\right) + C.}$$

Note that this can be rewritten as

$$\boxed{\int \frac{dx}{\sqrt{9x^2+16}} = \tfrac{1}{3}\ln\left(3x + \sqrt{16+9x^2}\right) + C.}$$

3a. Does $\sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^{3/2}}$ converge or diverge? Give your reasons.

Solution. We show that both

$$A = \sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}} \quad \text{and} \quad B = \sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$$

converge. This will imply that the original series converges. First note that

$$\left| \frac{\sin n}{n^{3/2}} \right| \le \frac{1}{n^{3/2}}.$$

It follows by direct comparison with the p-series for $p = 3/2$ that the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^{3/2}} \right|$$

converges. This shows that the series A converges (since the series obtained by taking the absolute value of each of its terms converges).

We now show that the series B converges. Recall that for any $a > 0$ we have $\ln n < n^a$ once n becomes sufficiently large. Taking $a = 1/4$ shows that we have the inequality

$$\frac{\ln n}{n^{3/2}} \le \frac{1}{n^{5/4}}.$$

The series B therefore converges by direct comparison with the p-series for $p = 5/4$.

Therefore, we have the final answer:

$$\boxed{\text{The series converges.}}$$

3b. Does the integral $\int_0^\infty \frac{\sin x}{x^2}\,dx$ converge or diverge? Give you reasons.

Solution. The integrand goes to infinity near zero. Therefore, zero and infinity are the places where this improper integral could possibly have problems. By definition, the original integral converges if and only if *both* of

$$\int_0^a \frac{\sin x}{x^2}\,dx \quad \text{and} \quad \int_a^\infty \frac{\sin x}{x^2}\,dx$$

converge, for some $0 < a < \infty$. We will take $a = 1$.

Consider the first integral. We have $\sin x \sim x$ as $x \to 0$. Therefore

$$\frac{\sin x}{x^2} \sim \frac{1}{x}$$

as $x \to 0$. Since

$$\int_0^1 \frac{1}{x}\,dx$$

diverges (by a version of the p-test) it follows from limit comparison that

$$\int_0^1 \frac{\sin x}{x^2}\,dx$$

diverges as well. Therefore, we obtain the answer:

The integral diverges.

4. Approximate the integral $\int_0^{1/10} \cos\sqrt{t}\, dt$ with an error less than 10^{-3}.

Solution. Recall the Taylor series for $\cos u$ about $u = 0$:

$$\cos u = 1 - \tfrac{1}{2}u^2 + \tfrac{1}{24}u^4 - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}.$$

Putting $u = \sqrt{t}$, we find

$$\cos\sqrt{t} = 1 - \tfrac{1}{2}t + \tfrac{1}{24}t^2 - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!}$$

We therefore have

$$\begin{aligned}
\int_0^{1/10} \cos\sqrt{t}\, dt &= \int_0^{1/10} \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!} \right) dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^{1/10} t^n\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{(1/10)^{n+1}}{n+1} \\
&= \tfrac{1}{10} - \tfrac{1}{400} + \tfrac{1}{24} \cdot \tfrac{10^{-3}}{4} - \ldots
\end{aligned}$$

We have expressed the integral we wish to approximate as an alternating series. It is clear that the terms of the alternating series are decreasing in absolute value. Therefore, if we use only the first two terms of the series the error is at most the absolute value of the third term. Since the absolute value of the third term is less than 10^{-3} it follows that using the first two terms of the series apporximates the integral well enough for the purposes of this problem. We therefore have our answer:

$$\boxed{\tfrac{1}{10} - \tfrac{1}{400} = \tfrac{39}{400} \text{ is an adequate approximation.}}$$

5. Find $\lim_{x \to 0} \dfrac{x \cos x - xe^{-x^2}}{\sin^3 x}$.

Solution. First of all, we have

$$\lim_{x \to 0} \frac{x \cos x - xe^{-x^2}}{\sin^3 x} = \lim_{x \to 0} \left[\left(\frac{x}{\sin x} \right)^3 \cdot \left(\frac{\cos x - e^{-x^2}}{x^2} \right) \right]$$

$$= \left(\lim_{x \to 0} \frac{x}{\sin x} \right)^3 \cdot \left(\lim_{x \to 0} \frac{\cos x - e^{-x^2}}{x^2} \right) = \lim_{x \to 0} \frac{\cos x - e^{-x^2}}{x^2}$$

since $\sin x/x \to 1$ as $x \to 0$. To do the remaining limit we can replace the numerator by the first term in its Taylor series expansion about $x = 0$. We have

$$\cos x = 1 - \tfrac{1}{2}x^2 + \dots$$
$$e^{-x^2} = 1 - x^2 + \dots$$

Therefore

$$\cos x - e^{-x^2} = \tfrac{1}{2}x^2 + \dots$$

and so

$$\lim_{x \to 0} \frac{\cos x - e^{-x^2}}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}x^2}{x^2} = \tfrac{1}{2}.$$

Therefore, we have our final answer:

$$\boxed{\lim_{x \to 0} \frac{x \cos x - xe^{-x^2}}{\sin^3 x} = \tfrac{1}{2}.}$$

6. Find all solutions, in Cartesian form, of $z^4 + 8iz = 0$.

Solution. First of all, $z = 0$ is a solution. The remaining solutions are thus solutions of $z^3 + 8i = 0$, or $z^3 = -8i$. We therefore only have to find the three cube roots of $-8i$. To do this, we write $-8i$ in polar form:

$$-8i = 8e^{\frac{3\pi i}{2} + 2\pi ik}$$

where k is any integer. We therefore see that the three cube roots of $-8i$ are given by

$$2e^{\frac{\pi i}{2} + \frac{2\pi ik}{3}}$$

for $k = 0, 1, 2$. We now just need to look at each value of these values of k and express the root in Cartesian form.

- $k = 0$. The exponent is then $\frac{\pi i}{2}$. We have $2e^{\pi i/2} = 2i$.
- $k = 1$. The exponent is then $\frac{\pi i}{2} + \frac{2\pi i}{3} = \frac{7\pi i}{6}$. We have $2e^{7\pi i/6} = -\sqrt{3} - i$.
- $k = 2$. The exponent if then $\frac{\pi i}{2} + \frac{4\pi i}{3} = \frac{11\pi i}{6}$. We have $2e^{11\pi i/6} = \sqrt{3} - i$.

We have therefore found all solutions to the original equation. Our final answer is:

The solutions to $z^4 + 8iz = 0$ are: 0, $2i$, $-\sqrt{3} - i$ and $\sqrt{3} - i$.

7. The region bounded by the curve $y = x^3 + 1$, the line $x = 0$ and the line $y = 9$ is revolved around the line $x = 3$. Find the volume.

Solution. We use the shell method. The curve meets the line $y = 9$ when $x = 2$. Therefore, we will have shells for x between 0 and 2. The radius of the shell at x is equal to $3 - x$ since we are revolving around $x = 3$. The height of the shell at x is equal to $9 - (x^3 + 1) = 8 - x^3$ since our region lies above $y = x^3 + 1$ and below $y = 9$. Therefore, the area of the shell at x is equal to $2\pi(3 - x)(8 - x^3)$. It follows that the volume of revolution is given by

$$\begin{aligned} \int_0^2 2\pi(3-x)(8-x^3)\,dx &= 2\pi \int_0^2 24 - 8x - 3x^3 + x^4\,dx \\ &= 2\pi\left(48 - 16 - 12 + \tfrac{32}{5}\right) \\ &= \tfrac{264}{5}\,\pi \end{aligned}$$

Our final answer is therefore:

$$\boxed{\text{The volume is equal to } \tfrac{264}{5}\,\pi.}$$

8. Find the length of the curve given in parametric form by $x = \sin^{-1}(t)$ and $y = \ln\sqrt{1-t^2}$ for $0 \le t \le \frac{1}{2}$.

Solution. The length L of the curve is given by

$$L = \int_0^{1/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

We have

$$x'(t) = \frac{1}{\sqrt{1-t^2}}, \qquad y'(t) = -\frac{t}{1-t^2}$$

and so

$$x'(t)^2 + y'(t)^2 = \frac{1}{1-t^2} + \frac{t^2}{(1-t^2)^2} = \frac{1}{(1-t^2)^2}.$$

Therefore

$$\sqrt{x'(t)^2 + y'(t)^2} = \frac{1}{1-t^2} = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)$$

and so

$$L = \frac{1}{2}\int_0^{1/2} \frac{1}{1+t} + \frac{1}{1-t}\, dt = \tfrac{1}{2}\left[\ln(1+t) - \ln(1-t)\right]_0^{1/2}$$

$$= \tfrac{1}{2}(\ln(3/2) - \ln(1/2)) = \tfrac{1}{2}\ln(3).$$

Our final answer is then:

$$\boxed{\text{The length of the curve is equal to } \tfrac{1}{2}\ln(3).}$$

9. Find all real solutions to the differential equation $x\frac{dy}{dx} + 2y = \sin x$.

Solution. We first divide by x to put the equation in a more familiar form:

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = \frac{\sin x}{x}. \tag{1}$$

The integrating factor for this first order linear equation is

$$\exp\left(\int \frac{2}{x}\, dx\right) = \exp(2\ln x) = x^2.$$

Multiplying both sides of (1) by x^2 gives the equation

$$\frac{d}{dx}\left(x^2 y\right) = x \sin x.$$

Integrating both sides of this gives

$$x^2 y = \int x \sin x\, dx. \tag{2}$$

To do the integral on the right side we integrate by parts with

$$u = x \qquad dv = \sin x\, dx$$
$$du = dx \qquad v = -\cos x.$$

This gives

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C.$$

Putting this in (2) gives

$$x^2 y = -x\cos x + \sin x + C.$$

Dividing by x^2 expresses y in terms of x, and gives us our final answer:

The solutions are given by $y = \dfrac{-x\cos x + \sin x + C}{x^2}$ where C is an arbitrary real constant.

10. Find all real solutions to the differential equation $y'' - 4y' + 8y = 16x^2$.

Solution. This is a second order equation with constant coefficients. Given any solution, all other solutions are gotten by adding solutions to the associated homogeneous equation. We therefore only need to solve the associated homogeneous equation and find one solution of the given equation.

We first find a solution of the given equation. Since the right side of the equation is a degree two polynomial, we can find a degree two polynomial which solves the equation. Therefore, let $y = ax^2 + bx + c$; we will find values of a, b and c which make this a soltion. We have

$$y'' = 2a \quad \text{and} \quad y' = 2ax + b.$$

Therefore,

$$\begin{aligned} y'' - 4y' + 8y &= (2a) - 4(2ax + b) + 8(ax^2 + bx + c) \\ &= (8a)x^2 + (8b - 8a)x + (8c - 4b + 2a). \end{aligned}$$

If this is to equal $16x^2$ then we must have

$$8a = 16, \quad 8b - 8a = 0, \quad 8c - 4b + 2a = 0.$$

This gives $a = 2$, $b = 2$ and $c = \frac{1}{2}$. Therefore

$$y = 2x^2 + 2x + \tfrac{1}{2} \tag{3}$$

is a solution to the equation.

We now solve the homogeneous equation. It is

$$y'' - 4y' + 8y = 0.$$

Its characteristic polynomial is $r^2 - 4r + 8$ whose roots are found to be $2\pi 2i$. It follows that the solutions of the homogeneous equation all have the form

$$Ae^{(2+2i)x} + Be^{(2-2i)x}$$

or, equivalently (by using Euler's formula),

$$Ae^{2x}\sin(2x) + Be^{2x}\cos(2x). \tag{4}$$

The general solution to the original equation is now given by adding the solutions (4) to the homogenous equation to the particular solution (3) to the original equation. This gives the final answer:

The solutions are given by $y = 2x^2 + 2x + \frac{1}{2} + Ae^{2x}\sin(2x) + Be^{2x}\cos(2x)$ where A and B are arbitrary real constants.