Solutions:

(1) Evaluate

$$\int \frac{x^2}{(1+x^2)^{3/2}}\,dx.$$

We use the trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta d\theta$

$$\begin{aligned}\int \frac{x^2}{(1+x^2)^{3/2}}\,dx &= \int \frac{\tan^2\theta\sec^2\theta}{\sec^3\theta}\,d\theta \\ &= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\ &= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta \\ &= \int \sec\theta\ d\theta - \int \cos\theta\ d\theta \\ &= \ln|\sec\theta + \tan\theta| - \sin\theta + C\end{aligned}$$

We have $\sin\theta = \frac{x}{\sqrt{x^2+1}}$ and $\cos\theta = \frac{1}{\sqrt{x^2+1}}$. Rewriting the above result in terms of x we get

$$\int \frac{x^2}{(1+x^2)^{3/2}}\,dx = \ln|\sqrt{x^2+1} + x| - \frac{x}{\sqrt{x^2+1}} + C.$$

(2) Evaluate

$$\int \frac{\ln(x^2+2x+2)}{(x+1)^2}\,dx.$$

We first make the change of variable $y = x + 1$, $dy = dx$ to get

$$\int \frac{\ln(x^2+2x+2)}{(x+1)^2}\, dx = \int \frac{\ln(y^2+1}{y^2}\, dy.$$

We then integrate by parts with $u = \ln(1+y^2)$, $du = \frac{2y}{1+y^2}dy$ and $dv = \frac{1}{y^2}dy$, $v = \frac{-1}{y}$

$$\begin{aligned}\int \frac{\ln(y^2+1)}{y^2}\, dy &= \frac{-\ln(1+y^2)}{y} + \int \frac{2}{1+y^2}\, dy \\ &= \frac{-\ln(1+y^2)}{y} + 2\tan^{-1} y + C.\end{aligned}$$

Rewriting in terms of the variable x

$$\int \frac{\ln(x^2+2x+2)}{(x+1)^2}\, dx = \frac{-\ln(1+(x+1)^2)}{x+1} + 2\tan^{-1}(x+1) + C$$

(3) Does $\int_2^\infty \frac{\ln(e^x-2)}{x^3+1}\, dx$ converge or diverge ?

The only trouble spot is at ∞. We have $\ln(e^x - 2) \sim x$ at $x \to \infty$ as

$$\lim_{x\to\infty} \frac{\ln(e^x-2)}{x} = 1$$

using L'Hospital rule. Also $x^3 + 1 \sim x^3$ at infinity. Therefore the convergence or divergence of the integral is the same as for the following integral

$$\int_2^\infty \frac{x}{x^3}\, dx = \int_2^\infty \frac{1}{x^2}\, dx.$$

By the p-test, this integral converges. Hence the above integral converges.

(4) a. Does $\sum_{n=0}^\infty \frac{3^n (n!)^2}{(2n)!}$ converge or diverge ?

We use the ratio test

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1}((n+1)!)^2}{(2(n+1))!} \frac{(2n)!}{3^n(n!)} \\ &= \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n\to\infty} \frac{3n^2+6n+3}{4n^2+6n+2} \\ &= \frac{3}{4} < 1\end{aligned}$$

By the ratio test the series converges.

b. Does $\sum_{n=1}^\infty \frac{e^{10n}+n^{10}}{n^n}$ converge or diverge ?

We split the series in two: $\sum_n \frac{e^{10n}}{n^n} + \sum_n \frac{n^{10}}{n^n}$. We use the root test on both. For the first, we get

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{e^{10n}}{n^n}\right)^{1/n} &= \lim_{n\to\infty} \left(\frac{e^{10}}{n}\right) \\ &= 0 < 1\end{aligned}$$

and

$$\lim_{n\to\infty}\left(\frac{n^{10}}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{(n^{1/n})^{10}}{n}\right)$$
$$= 0 < 1$$

Therefore both series converge so the sum of the two converges.

(5) For what values of x does $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^{1/2}}$ converge?

We first look at the interval of absolute convergence. Setting $a_n = \left|\frac{x^n}{n(\ln n)^{1/2}}\right|$. we have

$$\frac{a_{n+1}}{a_n} = |x|\frac{n(\ln n)^{1/2}}{(n+1)(\ln(n+1))^{1/2}}$$

We take the limit of the ratio $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ as

$$\lim_{n\to\infty}|x|\frac{n(\ln n)^{1/2}}{(n+1)(\ln(n+1))^{1/2}} = |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln n}{\ln n+\ln(1+1/n)}\right)^{1/2}$$
$$= |x|$$

where we have used $\ln(n+1) = \ln n + \ln(1+1/n)$. The ratio test that the series converges for $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$. In our case we have that the series converges absolutely for $|x| < 1$.

It remains to check the end point. If $x = -1$, the series reduces to $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^{1/2}}$. This is an alternating series. Moreover, $\frac{1}{n(\ln n)^{1/2}}$ is decreasing and converges to 0 as $n \to \infty$. By the alternating test, it must converge.

If $x = 1$, we get the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1/2}}.$$

This is a series with positive terms. The form of the term in the series suggests the use of the integral test. The relevant integral is

$$\int_2^{\infty}\frac{dx}{x(\ln x)^{1/2}}$$

We make the change of variable $v = \ln x$ to get

$$\int_2^{\infty}\frac{dx}{x(\ln x)^{1/2}} = \int_{\ln 2}^{\infty}\frac{dv}{v^{1/2}} = \infty$$

By the integral test we conclude that the series diverges at $x = 1$.

(6) Find

$$\lim_{x\to 0}\frac{e^{2x}-\cos x-\sin 2x}{\ln(1+x)-x}.$$

This a case $\frac{0}{0}$. We use Taylor series about $x = 0$. We have $e^{2x} = 1+2x+\frac{(2x)^2}{2!}+\ldots$, $\cos x = 1-\frac{x^2}{2!}+\ldots$, $\sin 2x = 2x-\frac{8x^3}{3!}+\ldots$ and

$$\ln(1+x) = \int_0^x \frac{1}{1+y}\,dy = \int_0^x (1-y+y^2+\ldots)\,dy = x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots$$

using the sum of a geometric series. The x^2-terms will be the dominant terms in the numerator and the denominator. Inserting the series, we get

$$\lim_{x\to 0}\frac{e^{2x}-\cos x-\sin 2x}{\ln(1+x)-x} = \lim_{x\to 0}\frac{(1+2x+4x^2/2+...)-(1-x^2/2+...)-(2x-...)}{(x-x^2/2+...)-x}$$
$$= \lim_{x\to 0}\frac{5x^2/2+...}{-x^2/2+...}$$
$$= -5$$

(7) Write $(1+i)^{15}(1+i\sqrt{3})^{17}$ in polar form with $r\geq$ and $0\leq\theta<2\pi$.

We first write each number in polar form. We have $1+i=\sqrt{2}e^{i\pi/4}$ as $r=\sqrt{1+1}$ and $\tan\frac{\pi}{4}=\frac{1/\sqrt{2}}{1/\sqrt{2}}=1$. The same way $1+i\sqrt{3}=2e^{i\pi/3}$. It is now easy to take powers:

$$(1+i)^{15} = 2^{15/2}e^{i\frac{15\pi}{4}} = 2^{15/2}e^{i\frac{7\pi}{4}}$$
$$(1+i\sqrt{3})^{17} = 2^{17}e^{i\frac{17\pi}{3}} = 2^{17}e^{i\frac{5\pi}{3}}$$

where we have used $\frac{15\pi}{4}=2\pi+\frac{7\pi}{4}$ and $\frac{17\pi}{3}=4\pi+\frac{5\pi}{3}$. It remains to take the product of both numbers

$$(1+i)^{15}(1+i\sqrt{3})^{17} = \left(2^{15/2}e^{i\frac{7\pi}{4}}\right)\left(2^{17}e^{i\frac{5\pi}{3}}\right) = 2^{49/2}e^{i\frac{41\pi}{12}} = 2^{49/2}e^{i\frac{17\pi}{12}}$$

as $\frac{41\pi}{12}=2\pi+\frac{17\pi}{12}$.

(8) Find all real solutions to the differential equations $\cos^2 x\frac{dy}{dx}+y=e^{\tan x}$.

This is a first-order linear equation. We divide by $\cos^2 x$ to get the usual form $\frac{dy}{dx}+\frac{1}{\cos^2 x}y=\frac{1}{\cos^2 x}e^{\tan x}$. Recall that $1/\cos^2 x=\sec^2 x$. By the form of the equation, the integrating factor is given by $e^{P(x)}$ where

$$P(x)=\int\sec^2 x\ dx=\tan x.$$

Multiplying the equation by the integrating factor $e^{\tan x}$ and using the product rule yields

$$\frac{d}{dx}\left(e^{\tan x}y\right)=\sec^2 x\ e^{2\tan x}.$$

Integration on both sides gives

$$e^{\tan x}y=\int\sec^2 x\ e^{2\tan x}\ dx=\frac{1}{2}e^{2\tan x}+C.$$

where we have used the change of variable $u=2\tan x$. We get the final answer by dividing by $e^{\tan x}$

(9) Find all real solutions to the differential equations $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=e^{3x}$.

The general solution is given by $y(x)=y_h(x)+y_p(x)$ where y_h is the solution to the homogeneous equation and y_p is the particular solution to the non-homogeneous equation.

We first find the solution to the homogeneous equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$. The characteristic polynomial of the equation is $r^2 + r - 2$. It has roots $r_1 = 1$ and $r_2 = -2$. Therefore we have

$$y_h(x) = C_1 e^x + C_2 e^{-2x}.$$

To find y_p, we guess from the equation that it should be of the form $y_p(x) = Ae^{3x}$ for some real constant A. To find A, we insert our guess in the equation to get

$$(9A + 3A - 2A)\, e^{3x} = e^{3x}.$$

Dividing by e^{3x}, we conclude that $A = 1/10$ for our guess to satisfy the equation. Therefore, the general solution to the equation is

$$y(x) = C_1 e^x + C_2 e^{-2x} + \frac{1}{10} e^{3x}.$$

(10) Find the volume of the solid obtained by revolving the region under the curve $y = \cos x$ and above the x-axis for $0 \le x \le \pi/3$ about the line $x = -1$.

We use the shell method. The shells have radius $x+1$ and height $\cos x$. Therefore the volume is given by

$$\begin{aligned} V &= \int_0^{\pi/3} 2\pi(x+1)\cos x \; dx \\ &= 2\pi \int_0^{\pi/3} x\cos x \; dx + 2\pi \int_0^{\pi/3} \cos x \; dx \\ &= 2\pi \left(x\sin x + \cos x\right)\Big|_0^{\pi/3} + 2\pi\left(\sin x\right)\Big|_0^{\pi/3} \\ &= 2\pi\left(\frac{\pi}{3}\frac{\sqrt{3}}{2} + \frac{1}{2} - 1\right) + 2\pi\frac{\sqrt{3}}{2} \\ &= \frac{\pi^2}{\sqrt{3}} + \pi(\sqrt{3} - 1) \end{aligned}$$

(11) Find the length of the curve given in the parametric form by

$$\begin{cases} x(t) = 2(t^2-1)^{3/2} \\ y(t) = 3t^2 \end{cases}$$

where $2 \le t \le 3$.

The length of the curve is given by the integral

$$L = \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

We have $\frac{dx}{dt} = 6t(t^2-1)^{1/2}$ and $\frac{dy}{dt} = 6t$. The integral becomes

$$\begin{aligned} L = \int_2^3 \sqrt{36t^2(t^2-1) + 36t^2} \; dt = 6\int_2^3 \sqrt{t^4} \; dt &= 6\int_2^3 t^2 \; dt \\ &= 6\left(\frac{t^3}{3}\right)\Big|_2^3 = 38 \end{aligned}$$