You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots$$

"Known" equations from geometry:

Volume of a sphere: $V = \frac{4}{3}\,\pi\, r^3$

Surface area of a sphere: $A = 4\pi\, r^2$

Volume of a cylinder: $V = \pi\, r^2\, h$

Volume of a cone: $V = \frac{1}{3}\,\pi\, r^2\, h$

1. [9 points] When a rocket leaves the gravitational influence of the Earth, it could travel infinitely far away (if we ignore the effects of other celestial bodies). When a rocket of mass m kilograms is h meters above the surface of the Earth, it has a weight of $w = 9.8m\left(\frac{6{,}400{,}000}{6{,}400{,}000+h}\right)^2$ Newtons. Here, $6,400,000$ is the radius of the Earth in meters, and 9.8 is the gravitational constant in $\mathrm{m/s^2}$.

a. [3 points] Approximately how much work is required to lift the rocket Δh additional meters when it is already h meters above the surface of the Earth? Your answer may include m, h, and Δh.

Solution: The weight of the rocket is $9.8m\left(\frac{6{,}400{,}000}{6{,}400{,}000+h}\right)^2$, and so the work needed is

$$9.8m\left(\frac{6,400,000}{6,400,000+h}\right)^2 \Delta h \text{ Joules.}$$

b. [6 points] Figure out the work required to lift the rocket from the surface of the Earth to a height of infinity. Your answer may include m.

Solution: We integrate as h goes between 0 and ∞ :

$$\begin{aligned}\int_0^\infty 9.8m\left(\frac{6,400,000}{6,400,000+h}\right)^2 dh &= 9.8m \lim_{b\to\infty}\int_0^b \left(\frac{6,400,000}{6,400,000+h}\right)^2 dh \\ &= 9.8m(6,400,000)^2 \lim_{b\to\infty}\int_{6{,}400{,}000}^{b+6{,}400{,}000} \frac{du}{u^2} \\ &= 9.8(6,400,000)^2 m \lim_{b\to\infty}\left[\frac{1}{b+6,400,000} - \frac{1}{6,400,000}\right] \\ &= 9.8(6,400,000)m \text{ Joules.}\end{aligned}$$

2. [18 points] For each of the following series, write whether the series "Converges" or "Diverges" on the space provided next to the series. Support your answer by stating the test(s) you used to prove convergence or divergence, and show complete work and justification.

a. [6 points] $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ ______________ **Diverges**

Solution: We note that $\frac{1}{n^2-1} > \frac{1}{n^2}$ and $\sqrt{n^2+1} > n$ for $n \geq 2$, so $\frac{\sqrt{n^2+1}}{n^2-1} > \frac{1}{n}$. We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test, so by the comparison test, the series diverges.

b. [6 points] $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ ______________ **Converges**

Solution: We use the ratio test to get

$$\lim_{n\to\infty} \frac{(2n)!(n+1)!(n+2)!}{(2(n+1))!(n+1)!n!}.$$

After some cancellation we get

$$\lim_{n\to\infty} \frac{(n+2)(n+1)}{(2n+1)(2n+2)} = 1/4.$$

Thus it converges.

c. [6 points] $\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$ ______________ **Converges**

Solution: We will show that this is absolutely convergent, and hence convergent. Additionally, since $|\sin(n)| \leq 1$ we can bound the sum of the absolute values by

$$\sum_{n=2}^{\infty} \frac{1}{n^2-3}.$$

Then we apply the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{n^2-3}{n^2} = 1$$

both series converge or both diverge. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-test, they both converge.

3. [12 points] For each of the following series, determine the interval of convergence and write it on the space provided to the right of the series. Be sure to show all appropriate work to justify your answer.

a. [6 points] $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n}$ $\underline{\qquad\qquad\qquad 1 < x \leq 3\ }$

Solution: By using the ratio test we see that

$$\lim_{n\to\infty} \frac{|x-2|^{n+1}(n)}{|x-2|^n(n+1)} = |x-2|.$$

So the center is at $x = 2$ and the radius is 1. We need to check the endpoints $x = 1$ and $x = 3$. We see it converges when $x = 3$ by alternating series, and diverges at $x = 1$ by the p-test.

b. [6 points] $\sum_{n=1}^{\infty} \frac{n! x^n}{n^{10}}$ $\underline{\qquad\qquad\qquad x = 0\ }$

Solution: By using the ratio test we get

$$\lim_{n\to\infty} \frac{(n+1)! x^{n+1} n^{10}}{n! x^n (n+1)^{10}} = \infty.$$

So it has radius of convergence 0 and converges only at the center which is $x = 0$.

4. [8 points] Consider a solid whose base is contained between the curves $y = e^x$, $y = 1$, and $x = 3$. Cross-sectional slices perpendicular to the x-axis are rectangles, having length contained in the base region mentioned above and height determined by $g(x) = x^2$. Determine the exact volume of this solid.

Solution: The slice has volume $x^2(e^x - 1)\Delta x$. Summing the slices and letting Δx go to 0, we have

$$\begin{aligned}
\text{Volume} &= \int_0^3 x^2(e^x - 1)dx \\
&= \int_0^3 x^2 e^x dx - \int_0^3 x^2 dx \\
&= (x^2e^x|_0^3 - \int_0^3 2xe^x dx) - \frac{1}{3}x^3|_0^3 \\
&= (x^2e^x|_0^3 - (2xe^x|_0^3 - \int_0^3 2e^x dx)) - \frac{1}{3}x^3|_0^3 \\
&= (x^2e^x - 2xe^x + 2e^x - \frac{1}{3}x^3)|_0^3 \\
&= 9e^3 - 6e^3 + 2e^3 - 9 - 2 \\
&= 5e^3 - 11
\end{aligned}$$

5. [8 points] Kyle buys a nine-ounce cup of coffee at the store 10 minutes before class every morning. He doesn't drink the coffee until he arrives at class, ten minutes later. Suppose the coffee's temperature is initially 180°F when he buys it.

a. [2 points] The temperature of the coffee in degrees Fahrenheit, T, at t minutes after it was purchased satisfies the differential equation $\frac{dT}{dt} = k(T - 70)$, where k is a constant and 70°F is the surrounding air's temperature. Determine the units of k.

Solution: Units of k are $\frac{1}{\text{min.}}$

b. [4 points] Solve the differential equation given in part (a), finding T as a function of t. Your answer may include the constant k.

Solution:

$$\begin{aligned} \frac{dT}{T-70} &= kdt \\ \int \frac{dT}{T-70} &= \int kdt \\ \ln|T-70| &= kt + C \\ T - 70 &= Ae^{kt} \\ T &= Ae^{kt} + 70 \end{aligned}$$

Since $T = 180$ when $t = 0$, we have $T = 110e^{kt} + 70$.

c. [2 points] Suppose that Kyle adds one ounce of 40°F milk upon arriving at class, ten minutes after the coffee was purchased. When one ounce of milk is mixed with nine ounces of coffee, the resulting mixture has the temperature equal to

$$\frac{1}{10}[(\text{Temp. of Milk}) + 9(\text{Temp. of Black Coffee})].$$

Determine the *exact* value of k if the temperature of the coffee-milk mixture is 100°F, immediately after the milk has been added.

Solution: The temperature of the milk is 40°F and the temperature of the coffee 10 minutes after purchase is $110e^{10k} + 70$. The temperature of the mixture is then $\frac{1}{10}[40 + 9(110e^{10k} + 70)] = \frac{1}{10}(670 + 990e^{10k})$, which equals 100°F. We solve for k:

$$\begin{aligned} \frac{1}{10}(670 + 990e^{10k}) &= 100 \\ 670 + 990e^{10k} &= 1000 \\ 990e^{10k} &= 330 \\ e^{10k} &= \frac{1}{3} \\ 10k &= \ln(1/3) \\ k &= \frac{1}{10}\ln\left(\frac{1}{3}\right) \end{aligned}$$

6. [10 points] Consider the function $f(x) = \ln(1+x)$ and its Taylor series about $x = 0$.

a. [4 points] Determine the first four non-zero terms of the Taylor series for $f(x) = \ln(1+x)$ about $x = 0$. Be sure to show enough work to support your answer.

Solution:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$$

b. [4 points] Find the first three non-zero terms of the Taylor series for $g(x) = \ln\left(\frac{1+x}{1-x}\right)$ about $x = 0$. Be sure to show enough work to support your answer. *(Hint: You may find it helpful to utilize properties of logarithms.)*

Solution: Using our answer from part (a), we have

$$\ln(1-x) = (-x) - \frac{1}{2}(-x)^2 + \frac{1}{3}(-x)^3 - \frac{1}{4}(-x)^4 + ... = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - ...$$

Since $g(x) = \ln(1+x) - \ln(1-x)$, we have

$$g(x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + ...$$

c. [2 points] Find the exact value of the sum of the series

$$2\left(\frac{3}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right)^3 + \frac{2}{5}\left(\frac{3}{4}\right)^5 + ... \ .$$

Solution: Using our answer for $g(x)$ is part (b), we see this is the sum for the Taylor series evaluated at $x = \frac{3}{4}$, so the sum of this series is $g\left(\frac{3}{4}\right) = \ln(7)$.

7. [11 points] Consider the differential equation $\frac{dy}{dx} = \frac{1}{3^{10x}}$.

a. [5 points] Let $C > 0$ be a small constant. Use Euler's method with step size $\Delta x = C$ to write an expression for $y(4C)$ given that $y(0) = 7.3$. Your answer may include C.

Solution:

x	y	$\frac{dy}{dx}$	Δy
0	7.3	1	C
C	$7.3 + C$	$\frac{1}{3^{10C}}$	$\frac{C}{3^{10C}}$
$2C$	$7.3 + C + \frac{C}{3^{10C}}$	$\frac{1}{3^{20C}}$	$\frac{C}{3^{20C}}$
$3C$	$7.3 + C + \frac{C}{3^{10C}} + \frac{C}{3^{20C}}$	$\frac{1}{3^{30C}}$	$\frac{C}{3^{30C}}$
$4C$	$7.3 + C + \frac{C}{3^{10C}} + \frac{C}{3^{20C}} + \frac{C}{3^{30C}}$		

We see $y(4C) \approx 7.3 + C + \frac{C}{3^{10C}} + \frac{C}{3^{20C}} + \frac{C}{3^{30C}}$.

b. [3 points] Write a closed form expression for the approximation of $y(nC)$, where n is a positive integer.

Solution:

$$y(nC) \approx 7.3 + C + \frac{C}{3^{10C}} + \frac{C}{3^{20C}} + \frac{C}{3^{30C}} + ... + \frac{C}{3^{10(n-1)C}}$$

$$y(nC) \approx 7.3 + C(1 + \frac{1}{3^{10C}} + \frac{1}{3^{20C}} + \frac{1}{3^{30C}} + ... + \frac{1}{3^{10(n-1)C}}$$

$$y(nC) \approx 7.3 + (C)\left(\frac{1 - \left(\frac{1}{3^{10C}}\right)^n}{1 - \frac{1}{3^{10C}}}\right)$$

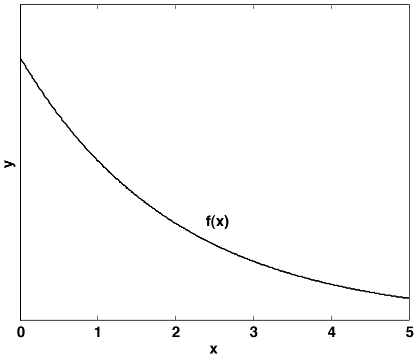
c. [3 points] Approximate $y(30)$ using $C = 0.1$.

Solution: We use our expression for $y(nC)$ in part (b), where $C = 0.1$ and $n = 300$. Then we have

$$y(30) \approx 7.3 + (0.1)\left(\frac{1 - \left(\frac{1}{3}\right)^{300}}{1 - \frac{1}{3}}\right) = 7.45$$

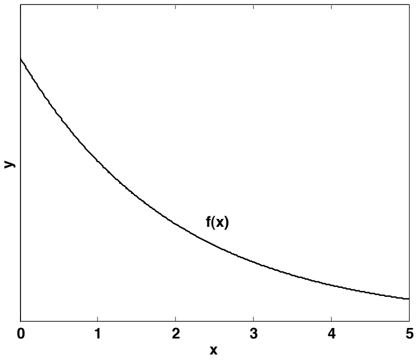
8. [12 points] For each of the following questions, draw a visual interpretation on the provided graph that would be useful in determining the specified quantity. Then write one complete sentence to explain your sketch.

a. [4 points] $\frac{f(4)-f(1)}{4-1}$



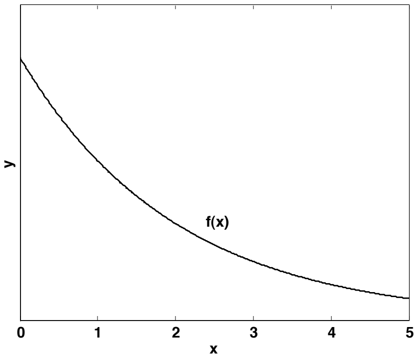
Solution: This is the slope of the secant line between $(1, f(1))$ and $(4, f(4))$.

b. [4 points] $\frac{1}{3}\int_1^4 f(x)dx$



Solution: This is the average value of $f(x)$ on $[1, 4]$.

c. [4 points] $(f(2) + f(3) + f(4))$



Solution: This is the RHS approximation for $\int_1^4 f(x)dx$. (LHS for $\int_2^5 f(x)dx$.)

9. [12 points] For each statement below, circle TRUE if the statement is *always* true; otherwise, circle FALSE. There is no partial credit on this page.

a. [2 points] If the power series $\sum C_n x^n$ converges at $x = 1$, then it converges at $x = -1$.

True [✓ False]

b. [2 points] Consider the point (x_0, y_0) given in Cartesian coordinates and its polar coordinates equivalent (r_0, θ_0). If $\frac{y_0}{x_0} = 1$, then $\theta_0 = \frac{\pi}{4}$.

True [✓ False]

c. [2 points] $\frac{d}{dx}\left(\int_x^2 \cos(\sin(t^2))dt\right) = \cos(\sin(x^2))$.

True [✓ False]

d. [2 points] Suppse $h(x)$ is a continuous function for $x > 0$. If $\int_1^\infty h(x)dx$ converges then for constant $0 < a < 1$, $\int_1^\infty h(\frac{x}{a})dx$ also converges.

[✓ True] False

e. [2 points] If $p(x)$ is a probability density function, then the units of $\int_{-\infty}^\infty xp(x)dx$ are the same as the units of x.

[✓ True] False

f. [2 points] The function $P(x) = (x-1) - \frac{1}{3!}(x-1)^3$ is the third degree Taylor polynomial for $f(x) = \sin(\pi x)$ about $x = 1$.

True [✓ False]