

1. (10 points) Let  $B$  be the rectangular solid  $0 \leq x \leq 4$ ,  $1 \leq y \leq 2$ ,  $0 \leq z \leq 2$ . Find

$$\iiint_B \frac{xz^2}{y^2} dV.$$

**SOLUTION:**  $\frac{xz^2}{y^2} = x y^{-2} z^2$ , so  $\iiint_B \frac{xz^2}{y^2} dV = \int_0^4 x dx \int_1^2 y^{-2} dy \int_0^2 z^2 dz =$   
 $= \left[ \frac{x^2}{2} \right]_{x=0}^4 \left[ -\frac{1}{y} \right]_{y=1}^2 \left[ \frac{z^3}{3} \right]_{z=0}^2 = \frac{32}{3}.$

2. (20 points) Let  $\vec{F}$  be the vector field

$$\vec{F}(x, y) = (y^2 + e^x) \vec{i} + 2xy \vec{j}.$$

- (a) (10 points) Find a real-valued function  $f(x, y)$  so that

$$\nabla f(x, y) = \vec{F}(x, y).$$

**SOLUTION:** We have  $\frac{\partial f}{\partial x} = y^2 + e^x$  and  $\frac{\partial f}{\partial y} = 2xy$ . The formula for  $\frac{\partial f}{\partial y}$  implies that  $f(x, y) = xy^2 + g(x)$  for **some** function  $g$  of one variable. Differentiating this formula with respect to  $x$  yields  $\frac{\partial f}{\partial x} = y^2 + g'(x)$ , so we need  $g'(x) = e^x$  and so  $g(x) = e^x + C$ . Therefore

$$f(x, y) = xy^2 + e^x + C.$$

- (b) (10 points) Let  $C$  be the curve given by  $x = \cos(t^2\pi)$  and  $y = t$ ,  $0 \leq t \leq 2$ . Find the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

**SOLUTION:** The curve  $C$  starts at  $x = 1$ ,  $y = 0$  and ends at  $x = \cos 4\pi = 1$ ,  $y = 2$ . So by the Fundamental Theorem of Calculus for Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(1, 2) - f(1, 0) = (4 + e + C) - (0 + e + C) = 4.$$

3. (15 points) Find the **surface area** of the portion of the paraboloid  $z = x^2 + y^2$  which lies below the plane  $z = 2$  (*Hint:* you might want to compute the double integral in polar coordinates.)

**SOLUTION:**  $A(S) = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA$ , where  $R$  is the circle  $x^2 + y^2 \leq 2$  of radius  $\sqrt{2}$ . So

$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2} dA = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2} r dr d\theta = \frac{2\pi}{8} \int_1^9 \sqrt{u} du =$$

$$\frac{\pi}{4} \left[ \frac{u^{3/2}}{3/2} \right]_1^9 = \frac{\pi}{6} (27 - 1) = \frac{13\pi}{3}.$$

4. (20 points) Under the linear transformation  $x = 4u + v$ ,  $y = 5u + 3v$  from the  $(u, v)$ -plane to the  $(x, y)$ -plane, the circular disk  $D$  given by the inequality  $u^2 + v^2 \leq 4$  is transformed into the elliptical region  $E$  given by  $34x^2 - 46xy + 17y^2 \leq 156$ . Compute the **area of  $E$  as an integral over  $D$** .

**SOLUTION:**  $D$  is a circular disk of radius 2, so  $A(D) = 4\pi$ . The Jacobian determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 4 \cdot 3 - 1 \cdot 5 = 7,$$

a constant. So  $A(E) = \iint_D 7 \, dA = 7A(D) = 28\pi$ .

5. (20 points) An oriented curve  $C$  in the  $(x, y)$ -plane consists of four pieces  $C_1, C_2, C_3, C_4$ :  $C_1$  is the upper semicircle from  $(2, 0)$  to  $(-2, 0)$ , given by  $x = 2 \cos t$ ,  $y = 2 \sin t$ ,  $0 \leq t \leq \pi$ ;  $C_2$  is the segment of the  $x$ -axis from  $(-2, 0)$  to  $(-1, 0)$ ;  $C_3$  is the upper semicircle from  $(-1, 0)$  to  $(1, 0)$ , given by  $x = -\cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq \pi$ ; and  $C_4$  is the segment of the  $x$ -axis from  $(1, 0)$  to  $(2, 0)$ . Let  $\vec{F}$  be the vector field

$$\vec{F}(x, y) = [e^{x^2} + \sin y] \vec{i} + [x \cos y + 3x + \ln(y + 1)] \vec{j}.$$

Find  $\int_C \vec{F} \cdot d\vec{r}$ . (*Hint:* Green's Theorem makes this much easier! You may use what you remember about areas inside circles.)

**SOLUTION:** The curve  $C$  is the oriented boundary of a region  $D$  which can be described as the upper semi-circle of radius 2 centered at  $(0, 0)$  minus the semi-circle of radius 1 centered at  $(0, 0)$ . In polar coordinates,  $D$  is described by  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq \pi$ . Write  $\vec{F}(x, y) = P\vec{i} + Q\vec{j}$ , so  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y + 3 - \cos y = 3$ . By Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA = \int_0^\pi \int_1^2 3r \, dr \, d\theta = 3\pi \left[ \frac{r^2}{2} \right]_1^2 = \frac{9\pi}{2}.$$

6. (15 points) The portion of the ball of radius 2 with center at  $(0, 0, 0)$ , which lies above the cone

$$z = \frac{1}{2} \sqrt{x^2 + y^2 + z^2},$$

is described in spherical coordinates by  $0 \leq \rho \leq 2$ ,  $0 \leq \phi \leq \pi/3$ ,  $0 \leq \theta \leq 2\pi$ . Find the **volume** of this figure by computing an integral in spherical coordinates. (*Hint:* Recall that  $\sin \pi/3 = \frac{1}{2}\sqrt{3}$  and  $\cos \pi/3 = \frac{1}{2}$ .)

**SOLUTION:** The volume is  $V =$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/3} \sin \phi \, d\phi \int_0^2 \rho^2 \, d\rho = 2\pi [-\cos \phi]_0^{\pi/3} \cdot i \left[ \frac{\rho^3}{3} \right]_0^2 = \frac{8\pi}{3}.$$