Solutions

Problem 1. *(10 pts.)*

(a – 5 pts.) Find the area of the parallelogram spanned by the vectors $\mathbf{u} = \langle 3, 2, -1\rangle$ *and* $\mathbf{v} = \langle 1, -2, 3\rangle$.

Solution. By the properties of the *cross-product*,

$$\text{Area}(\mathbf{u}, \mathbf{v}) = |\mathbf{u} \times \mathbf{v}| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & -2 & 3 \end{matrix} \right\| = |\langle 4, -10, -8\rangle| = \sqrt{4^2 + (-10)^2 + (-8)^2} = \sqrt{180} = 6\sqrt{5}$$

♣

(b – 5 pts.) Find the rate of change of the function $f(x, y) = x^2y + y^3$ *at the point* $(2, 1)$ *in the direction of the vector* $\mathbf{v} = \langle 3, -2\rangle$.

Solution. We have:

$$\text{Rate of change} = D_{\mathbf{u}} f(2, 1) = \text{comp}_{\mathbf{v}} \nabla f(2, 1) = \nabla f(2, 1) \bullet \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \langle 2xy, x^2 + 3y^2\rangle\big|_{\substack{x=2\\y=1}} \bullet \frac{\langle 3, -2\rangle}{|\langle 3, -2\rangle|} = \langle 4, 7\rangle \bullet \frac{\langle 3, -2\rangle}{\sqrt{13}} = \frac{-2}{\sqrt{13}},$$

so the function is decreasing in this direction. ♣

Problem 2. *(10 pts.) Three quantities* x, y, *and* z *are related by the equation* $x^3ze^{z^2-y^2} = 2$. *First find the equation of the tangent plane to the surface defined by this equation at the point* $(1, -2, 2)$ *and then use it to approximate the value of* z *corresponding to* $x = 1.2$ *and* $y = -2.1$.

Solution. To find the equation of the level surface defined by the equation $F(x, y, z) = x^3ze^{z^2-y^2} = 2$ we use the property that ∇F is perpendicular to levels of F. Then

$$\nabla F = \langle 3x^2ze^{z^2-y^2}, -2x^3yze^{z^2-y^2}, x^3e^{z^2-y^2} + 2x^3z^2e^{z^2-y^2}\rangle \qquad \nabla F(1, -2, 2) = \langle 6, 8, 9\rangle$$

and we get a normal vector $\mathbf{n} = \langle 6, 8, 9\rangle$ to the tangent plane. Then the equation of the tangent plane is

$$6(x - 1) + 8(y + 2) + 9(z - 2) = 0.$$

Solving it for z we get an approximation for $z = z(x, y)$ near $(1, -2, 2)$:

$$z \approx 2 - \frac{6(x - 1) + 8(y + 2)}{9} = 2 - \frac{6\Delta x + 8\Delta y}{9}$$

$$z(1.2, -2.1) \approx 2 - \frac{6 \cdot 0.2 + 8 \cdot (-0.1)}{9} = 2 - \frac{0.4}{9} = 2 - 0.0\bar{3} \approx 1.97.$$

♣

Problem 3. *(10 pts.) Find the mass of a cylindrical surface of radius* $r = 3$ *centered on the* z*-axis and bounded by the planes* $z = 1$ *and* $z = 3$ *if the density function is equal to the distance to the* xy*-plane.*

Solution. The density function $\sigma(x, y, z) = z$, and so Mass $= \iint_S \sigma\, dS$. We parameterize our cylindrical surface S by

$$\mathbf{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z\rangle, \qquad 0 \le \theta \le 2\pi, \quad 1 \le z \le 3,$$

then

$$dS = |\mathbf{r}_\theta \times \mathbf{r}_z|\, d\theta dz = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{matrix} \right\| d\theta dz = |\langle 3\cos\theta, 3\sin\theta, 0\rangle| = 3d\theta dz,$$

$$\text{Mass} = \iint_S \sigma\, dS = \int_1^3 \int_0^{2\pi} z \cdot 3\, d\theta dz = 3\pi z^2\Big|_1^3 = 24\pi.$$

♣

Problem 4. *(10 pts.) Using any method you like (for example, the distance formula, Lagrange multipliers, etc.) find the distance from the point $A(17,-4,-3)$ to the point on the plane $6x-3y+2z=10$ closest to A. What is this point?*

Solution.

The distance formula To use the distance formula, we first find some point on the plane, for example $P(1,0,2)$. Then the vector $\vec{\mathbf{PA}} = \langle 16,-4,-5\rangle$. We also need a normal vector to the plane, we take it to be $\mathbf{n} = \langle 6,-3,2\rangle$. Then

$$\operatorname{comp}_{\mathbf{n}} \vec{\mathbf{PA}} = \frac{\langle 16,-4,-5\rangle \bullet \langle 6,-3,2\rangle}{|\langle 6,-3,2\rangle|} = \frac{96+12-10}{\sqrt{49}} = \frac{98}{7} = 14, \qquad d = |\operatorname{comp}_{\mathbf{n}} \vec{\mathbf{PA}}| = 14$$

We can then get the closest point B by

$$\vec{OB} = \vec{OA} - \mathbf{proj}_n \vec{\mathbf{PA}} = \langle 17,-4,-3\rangle - 14\frac{\langle 6,-3,2\rangle}{4} = \langle 17,-4,-3\rangle - \langle 12,-6,4\rangle = \langle 5,2,-7\rangle.$$

Equations of Lines and Planes Alternatively, we can find parametric equations of the line going through the point A in the direction of the normal vector to the plane $\mathbf{n} = \langle 6,-3,2\rangle$:

$$\mathbf{l}(t) = \langle 17+6t, -4-3t, -3+2t\rangle,$$

find the point of intersection of this line with the plane by solving for t the equation

$$6x-3y+2z = 6(17+6t) - 3(-4-3t) + 2(-3+2t) = 108+49t = 10 \qquad \Longrightarrow \qquad t=-2$$

and then plugging it in:

$$\mathbf{l}(2) = \langle 17-12, -4+6, -3-4\rangle = \langle 5,2,-7\rangle = \vec{OB}.$$

We can then find the distance using the distance formula for points:

$$d = \operatorname{dist}(A,B) = \sqrt{(17-5)^2 + (-4-2)^2 + (-3+7)^2} = \sqrt{196} = 14.$$

Lagrange Multipliers Take our goal function $f(x,y,z) = \operatorname{dist}(A, P(x,y,z))^2$. Then we need to solve the following constrained optimization problem:

$$f(x,y,z) = (x-17)^2 + (y+4)^2 + (z+3)^2 \to \max/\min \qquad \text{subject to the constraint}$$
$$g(x,y,z) = 6x-3y+2z = 10.$$

The Lagrange multiplier equations then are:

$$\begin{cases} \nabla f = 2\langle x-17, y+4, z+3\rangle = \lambda \nabla g = \lambda\langle 6,-3,2\rangle \\ g = 6x-3y+2z = 10 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x = \frac{6\lambda}{2} + 17 \\ y = -\frac{3\lambda}{2} - 4 \\ z = \frac{2\lambda}{2} - 3 \\ 6x-3y+2z = 10 \end{cases}$$

$$6\left(\frac{6\lambda}{2} + 17\right) - 3\left(-\frac{3\lambda}{2} - 4\right) + 2\left(\frac{2\lambda}{2} - 3\right) = 10 \qquad \Longrightarrow \qquad \lambda = -4,$$

and as before $x=5$, $y=2$, $z=-7$, $d = \sqrt{f(5,2,-7)} = 14$. ♣

Problem 5. *(10 pts.)*

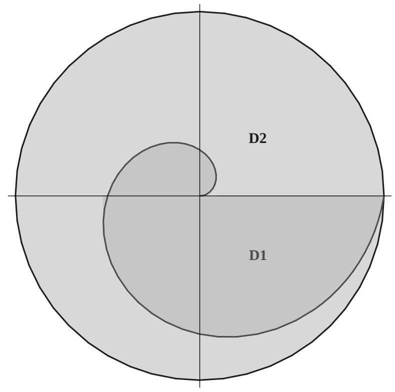
The great Greek mathematician Archimedes *in his treatise* On Spirals† *gave the following definition of a spiral:*

> *If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line, beginning from the extremity which remains fixed, the point will describe a spiral in the plane.*

Proposition 24 *in this treatise states that*

> *The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle'.*

Archimedes proved it using the method of exhaustion.
Use integration in polar coordinates to give a simple proof of this Proposition. *Note that in polar coordinates the equation of the* Archimedian spiral *takes a simple form $r = a\theta$, where a is some constant. In the notations of the picture on the right, you have to show that Area(D_1) = $\frac{1}{3}$Area(D_2) (note that D_1 is contained in D_2).*



† *The Works of Archimedes*, **Dover**, 1897

Solution. First note that the radius of the 'first circle' is $r(2\pi) = 2\pi a$. Therefore,

$$\text{Area}(D_2) = \pi(2\pi a)^2 = 4\pi^3 a^2$$

$$\text{Area}(D_1) = \iint_{D_1} 1\, dA = \int_0^{2\pi}\int_0^{a\theta} r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^{a\theta} d\theta = \frac{a^2}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{a^2}{2}\frac{(2\pi)^3}{3} = \frac{4a^2\pi^3}{3} = \frac{1}{3}\text{Area}(D_2).$$

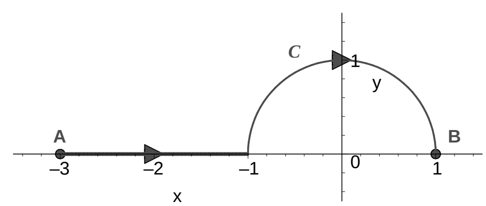
♣

Problem 6. *(15 pts.)*

In this problem we consider two vector fields,

$$\mathbf{F} = \langle -y, x\rangle, \qquad \textit{and} \qquad \mathbf{G} = \langle \cos(x) + y, x - 1\rangle,$$

and the curve C from the point $A(-3, 0)$ to the point $B(1, 0)$ that first goes along the x-axis, and then follows the unit circle (see the picture on the right).



(a – 5 pts.) Carefully explain whether these vector fields are conservative or not.

Solution. We begin by computing the curl:
curl$(\mathbf{F}) = Q_x - P_y = 1 - (-1) = 2 \neq 0$, so $\mathbf{F}$ can not be a conservative vector field.
curl$(\mathbf{G}) = 1 - 1 = 0$, so $\mathbf{G}$ *can* be a conservative vector field $\nabla f = \langle f_x, f_y\rangle$. Let us try to find a potential function $f(x, y)$. We know that then we must have $f_x = \cos(x) + y$ and $f_y = x - 1$. Then $f(x, y) = \sin(x) + xy + g(y) = xy - y + h(x)$. 'Merging' both expressions we get $f(x, y) = \sin(x) + xy - y + C$, and we'll take $C = 0$. Since we have shown that there exists $f(x, y)$ such that $\mathbf{G} = \nabla f$, $\mathbf{G}$ is a conservative vector field. ♣

(b – 5 pts.) Compute the work done by the field $\mathbf{F}$ along the curve C.

Solution. Since $\mathbf{F}$ is not conservative, we need to compute $\int_C \mathbf{F} \bullet d\mathbf{r}$ using parameterization. We split C in two pieces:

C_1 :	$\mathbf{r}(t) = \langle t, 0\rangle$,	$-3 \le t \le -1$	the piece along the x-axis
C_2 :	$\mathbf{r}(t) = \langle -\cos(t), \sin(t)\rangle$,	$0 \le t \le \pi$	the piece along the unit circle,

where the minus sign in the x-component of C_2 reflects the fact that it goes counterclockwise. Then

$$\int_C \mathbf{F} \bullet \mathbf{dr} = \int_{C_1} \mathbf{F} \bullet \mathbf{dr} + \int_{C_2} \mathbf{F} \bullet \mathbf{dr} = \int_{-3}^{-1} \langle 0, t\rangle \bullet \langle 1, 0\rangle \, dt + \int_0^{\pi} \langle -\sin(t), -\cos(t)\rangle \bullet \langle \sin(t), \cos(t)\rangle \, dt = 0 - \pi = -\pi.$$

♣

(c – 5 pts.) Compute the work done by the field **G** *along the curve* C.

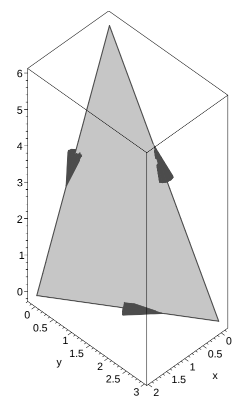
Solution. Since **G** is conservative, we can use the **Fundamental Theorem for Line Integrals**:

$$\int_C \mathbf{G} \bullet \mathbf{dr} = \int_C \nabla f \bullet \mathbf{dr} = f(x,y)\Big|_{(-3,0)}^{(1,0)} = f(1,0) - f(-3,0) = (\sin(1) - \sin(-3)) = \sin(1) + \sin(3).$$

♣

Problem 7. *(10 pts.)*

Find the circulation of the vector field $\mathbf{F} = (4xy + xz)\mathbf{i} + (xy - yz)\mathbf{j} + (z^2 - xz)\mathbf{k}$ *along the curve* C *which is the boundary of the triangle with vertices* $A(2, 0, 0)$, $B(0, 3, 0)$, *and* $C(0, 0, 6)$ *oriented as shown.*
*(*Suggestion*: Use* ***Stokes' Theorem****).*



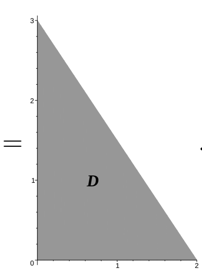
Solution. By **Stokes' Theorem**,

$$\oint_C \mathbf{F} \bullet \mathbf{dr} = \iint_S \operatorname{curl}(\mathbf{F}) \bullet \mathbf{dS},$$

where

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy + xz & xy - yz & z^2 - xz \end{vmatrix} = \langle 0 + y, x + z, y - 4x\rangle,$$

and S is the surface of the triangle $\triangle ABC$ that is a graph of the function

$$z = 6 - 3x - 2y, \quad \text{where} \quad (x, y) \in D =$$  .

Then we can use the formula for the special case of the graphs of functions to compute the surface area element **dS**,

$$\mathbf{dS} = \langle -f_x, -f_y, 1\rangle \, dA = \langle 3, 2, 1\rangle \, dA$$

$$\oint_C \mathbf{F} \bullet \mathbf{dr} = \iint_S \operatorname{curl}(\mathbf{F}) \bullet \mathbf{dS} = \iint_D \langle y, x + (6 - 3x - 2y), y - 4x\rangle \bullet \langle 3, 2, 1\rangle \, dA = \iint_D (12 - 8x) \, dA$$

$$= \int_0^2 \int_0^{3 - \frac{3}{2}x} (12 - 8x) \, dy \, dx = 12 \cdot \text{Area}(D) - 8 \int_0^2 x \left(3 - \frac{3}{2}x\right) \, dx$$

$$= 12 \cdot \frac{1}{2} \cdot 2 \cdot 3 - 8 \left(\frac{3}{2}x^2\Big|_0^2 - \frac{1}{2}x^3\Big|_0^2 \right) = 36 - 8(6 - 4) = 20.$$

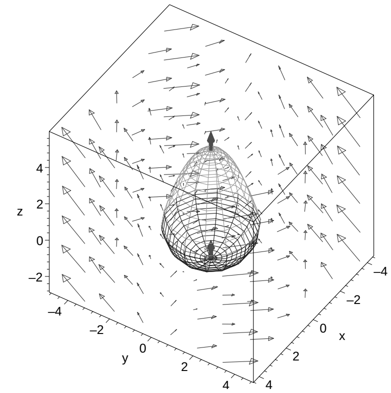
♣

Problem 8. *(15 pts.)*

*Use the **Divergence Theorem** to compare (by evaluating the difference) the flux of the vector field*

$$\mathbf{F} = (-y^2 + \cos(z))\mathbf{i} + (3xy + \sin(z))\mathbf{j} + (z + x^2)\mathbf{k}$$

through the surfaces S_1 and S_2, where S_1 is a lower hemisphere of radius 2 and S_2 is a part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane and both surfaces are oriented upward (see the picture).



Solution. By the **Divergence Theorem**, $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S} - \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F})\, dV$, where E is the region bounded by S_1 and S_2 and $\operatorname{div}(\mathbf{F}) = P_x + Q_y + R_z = 0 + 3x + 1$. So

$$\iint_{S_2} \mathbf{F} \bullet d\mathbf{S} - \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iiint_E (3x + 1)\, dV = 0 + \iiint_E 1\, dV = \text{Volume}(E),$$

where $\iiint_E x\, dV = 0$ by symmetry, and Volume(E) can be easily computed in cylindrical coordinates:

$$\begin{aligned}\text{Volume}(E) &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{4-r^2} 1 \cdot r\, dz\, dr\, d\theta = 2\pi \int_0^2 r(4 - r^2 - \sqrt{4 - r^2})\, dr \\ &= 2\pi \left(2r^2 \Big|_0^2 - \frac{1}{4} r^4 \Big|_0^2 + \frac{2 \cdot (4 - r^2)^{\frac{3}{2}}}{3 \cdot (-2)} \Big|_0^2 \right) = 2\pi \left(8 - 4 + \frac{8}{3} \right) = \frac{40\pi}{3}.\end{aligned}$$

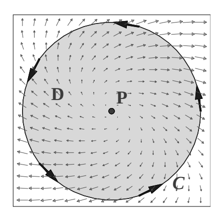
Therefore, the flux through S_2 is bigger than the flux through the S_1 by $\frac{40\pi}{3}$. ♣

Problem 9. *(10 pts.) Below are six pictures of a vector field* $\mathbf{F}$*, region D and its oriented boundary C, and a point P inside D. For each of the given properties, indicate all plots that have that property.*

Solution.

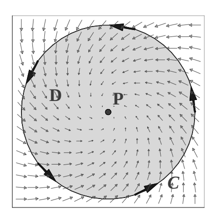
We can see that P is a source, so $\operatorname{div}(\mathbf{F})(P) > 0$ (by continuity, the same is true for nearby points as well), flux of $\mathbf{F}$ across C is positive, field spins clockwise, so $\operatorname{curl}(\mathbf{F}) < 0$, and circulation of $\mathbf{F}$ around C is negative (and therefore $\mathbf{F}$ can not be a gradient vector field).

(A)



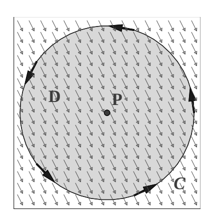
We can see that P is a sink, so $\operatorname{div}(\mathbf{F})(P) < 0$, flux of $\mathbf{F}$ across C is negative, field spins counterclockwise, so $\operatorname{curl}(\mathbf{F}) > 0$, and circulation of $\mathbf{F}$ around C is positive, $\mathbf{F}$ can not be a gradient vector field.

(B)



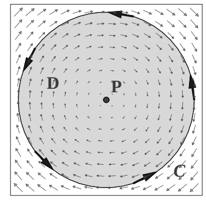
This field looks like a constant vector field. Constant fields have no curl or divergence and are gradient fields.

(C)



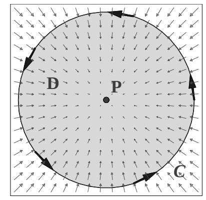
This field looks like a pure rotational field, so its curl is negative (since it spins clockwise), circulation around C is negative, divergence and flux are zero. This field can not be a gradient vector field (fields that have no divergence are also called *incompressible*).

(D)



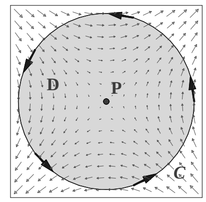
We can see that P is a sink, so $\text{div}(\mathbf{F})(P) < 0$, and the flux of $\mathbf{F}$ across C is negative. This field clearly does not spin, so its curl and circulation are zero (fields that have no circulation are also called *irrotational*). This field looks like it can be gradient (it actually is).

(E)



By symmetry it is clear that both circulation of $\mathbf{F}$ around C and flux of $\mathbf{F}$ across C are zero (contributions from different parts of the curve cancel each other out). This is also true for any small circle around P, so $\text{curl}(\mathbf{F})(P) = 0$ and $\text{div}(\mathbf{F})(P) = 0$. Since $\text{curl}(\mathbf{F}) = 0$ and it looks like it is defined everywhere, it can be a gradient vector field (and again it is).

(F)



So we get:

(a) $\text{curl}(\mathbf{F})(P) > 0$ **B**

(b) Circulation of $\mathbf{F}$ around C is positive **B**

(c) Circulation of $\mathbf{F}$ around C is negative **A,D**

(d) $\text{div}(\mathbf{F})(P) > 0$ **A**

(e) Flux of $\mathbf{F}$ across C is negative **B,E**

(f) $\mathbf{F}$ can be a gradient vector field **C,E,F**

♣