Problem 1. TRUE FALSE QUESTIONS. NO PARTIAL CREDIT. CIRCLE TRUE OR FALSE. IF YOU THINK A QUESTION DOESN'T MAKE SENSE, CIRCLE FALSE

(a)

$$\int \frac{1}{t^2}dt = \frac{-2}{t^3} + C \quad TRUE/FALSE$$

This is derivative, not integral. **FALSE**

(b)

$$\frac{d}{dx}(3 \arctan 5x) = \frac{3}{1+(5x)^2} \qquad TRUE/FALSE$$

Need to use the chain rule. **FALSE**

(c) The dot product of two vectors is always orthogonal to the plane through the two vectors. TRUE/FALSE
Dot product is a number, not a vector.
FALSE

(d) The two expressions $\vec{r}_1(t) = (\cos t, \sin t, t)$ and $\vec{r}_2(t) = (\cos(2t), \sin(2t), 2t)$ parametrize the same helix. TRUE/FALSE
Replacing t by $2t$ reparametrizes the curve.**TRUE**

(e)

$$\int_1^2 \sqrt{t^4 + 2 + 1/t^4}dt = \frac{17}{6} \qquad TRUE/FALSE$$

TRUE

$$\begin{aligned} \int_1^2 \sqrt{t^4 + 2 + 1/t^4}dt &= \int_1^2 \sqrt{(t^2 + 1/t^2)^2}dt \\ &= \int_1^2 t^2 + 1/t^2 dt \\ &= t^3/3 - 1/t|_2^2 \\ &= (8/3 - 1/2) - (1/3 - 1) \\ &= 7/3 + 1/2 = 17/6 \end{aligned}$$

(f) The gradient of the unit normal vector to a surface is always tangent to the surface. TRUE/FALSE

The unit normal is a vector, not a function.

FALSE

WORKSPACE:

Problem 2. NO PARTIAL CREDIT Consider the points $A(1,\ 0,\ 1)$, $B(2,\ -1,\ 1)$, $C(1,\ 2,\ 0)$.

(a) Find $\vec{AB} \cdot \vec{AC}$.

(b) Find the COSINE of the angle between $\vec{AB}$ and $\vec{AC}$.

(c) Find $\vec{AB} \times \vec{AC}$

(d) Find the SINE of the angle between $\vec{AB}$ and $\vec{AC}$.

(e) Find an equation for the plane P through the points A, B, C.

(f) Find a vector equation **or** a parametric equation for the line perpendicular to P through the point A.

NOTE: Go over your calculations again to make sure you have the right expressions for $\vec{AB}, \vec{AC}$. You lose a lot of points if you get them wrong.

ANSWERS:
a: -2
b: $\cos\theta = \frac{-2}{\sqrt{2}\sqrt{5}} = -\sqrt{2/5}$
c: $< 1, 1, 2 >$
d: $\sqrt{3/5}$
e: $x + y + 2z = 3$
f: $\vec{r}(t) =< 1 + t, t, 1 + 2t >, x = 1 + t, y = t, z = 1 + 2t$

WORKSPACE: $\vec{AB} =< 1, -1, 0 >, \vec{AC} =< 0, 2, -1 >$
a) $\vec{AB} \cdot \vec{AC} = -2$
b) $\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{-2}{\sqrt{2}\sqrt{5}}$
c)...
d) $\sin\theta = \frac{|\vec{AB}\times\vec{AC}|}{|\vec{AB}||\vec{AC}|} = \frac{\sqrt{6}}{\sqrt{2}\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}$
e) $< 1, 1, 2 > \cdot < x - 1, y, z - 1 >= x + y + 2z - 3 = 0.$
f) $\vec{r}(t) =< 1, 0, 1 > +t < 1, 1, 2 >=< 1 + t, t, 1 + 2t >$.

Problem 3. NO PARTIAL CREDIT. THE THREE PARTS OF THE PROBLEM ARE UNRELATED

(a) Find a vector equation **or** a parametric equation for the tangent line of the curve $x = t^4$, $y = 2\ln t$, $z = e^{2t}$ at the point $(1, 0, e^2)$

(b) Suppose a particle travels along the helix $< \cos(2t), \sin(2t), 3t >$ from the point $(1, 0, 0)$ through 5 complete turns of the helix. What is the total distance travelled by the particle along the helix?

(c) Suppose z is defined implicitly as a function of x, y by

$$x^2 e^y z = x\cos(y) + \cos(z).$$

Find $\frac{\partial z}{\partial y}$. Your answer may involve x, y and z.

ANSWERS:
a: $< 1 + 4s, 2s, e^2 + 2se^2 >$
b: $5\sqrt{13}\pi$
c: $z_y = \frac{-x\sin y - x^2 e^y z}{x^2 e^y + \sin z}$

WORKSPACE:
a) $t = 1$, $\vec{r}'(t) =< 4t^3, 2/t, 2e^{2t} >=< 4, 2, 2e^2 >$ Tangent line: $< 1 + 4s, 2s, e^2 + 2se^2 >$

b) $0 \leq t \leq 5\pi$.

$$\begin{aligned} \int_0^{5\pi} \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + 3^2} dt &= \int_0^{5\pi} \sqrt{13} \\ &= 5\sqrt{13}\pi \end{aligned}$$

c)

$$\begin{aligned} x^2 e^y z + x^2 e^y z_y &= -x\sin y - (\sin z) z_y \\ z_y(x^2 e^y + \sin z) &= -x\sin y - x^2 e^y z \\ z_y &= \frac{-x\sin y - x^2 e^y z}{x^2 e^y + \sin z} \end{aligned}$$

Problem 4. IF YOUR ANSWER IS CORRECT YOU GET FULL SCORE. IF ANSWER IS WRONG YOU CAN GET PARTIAL CREDIT IF YOU EXPLAIN YOURSELF CAREFULLY

(a) Let $f(x,y) = xye^{-xy^2}$. Find the gradient of f

(b) Find the directional derivative of $f(x,y) = xye^{-xy^2}$ at the point $(1,1)$ in the direction $< 2/\sqrt{5}, 1/\sqrt{5} >$.

(c) Find the equation of the tangent plane to the surface $z = f(x,y) = xye^{-xy^2}$ at the point $(1,1,e^{-1})$

(d) If $z = f(x,y) = xye^{-xy^2}$ and $x = u^2 + 3v,\ y = uv - 3v$, find $\frac{\partial z}{\partial v}$ at the point $u = 2, v = -1$.

ANSWERS:
a: $\nabla f =< ye^{-xy^2} - xy^3e^{-xy^2}, xe^{-xy^2} - 2x^2y^2e^{-xy^2} >$
b: $\frac{-1}{e\sqrt{5}}$
c: $z = 2/e - y/e$
d: $1/e$
WORKSPACE:
b) $\nabla f =< 0, -1/e >$.
c) $z = e^{-1} + f_x(x-1) + f_y(y-1) = e^{-1} - e^{-1}(y-1)$.
d)

$$x(2,-1) = 1, y(2,-1) = 1$$

$$\begin{aligned} z_v &= f_x x_v + f_y y_v \\ &= -e^{-1}(u-3) = 1/e \end{aligned}$$

Problem 5. IF YOUR ANSWER IS CORRECT YOU GET FULL SCORE. IF YOUR ANSWER IS WRONG YOU CAN GET PARTIAL CREDIT IF YOU EXPLAIN YOURSELF CAREFULLY

(a) Find the local maxima, local minima and saddle points of the function $f(x, y) = y^3 - 6xy + 8x^2$.
(b) Find the maximum and minimum values of the function $f(x, y) = xy^2$ subject to the constraint $x^2 + y^2 = 4$.
ANSWERS:
a: $(0, 0)$ saddle, $(9/32, 3/4)$ local minimum.
b: $\pm\frac{16}{3\sqrt{3}}$
WORKSPACE: a) $f_x = -6y + 16x, f_y = 3y^2 - 6x$

$$\begin{aligned} -6y + 16x = 0 &\qquad 3y^2 - 6x = 0 \\ y = 8x/3 &\qquad 3(8x/3)^2 - 6x = 0 \\ 64x^2 = 18x &\qquad x^2 = 9x/32 \\ (x, y) = (0, 0) &\qquad (x, y) = (9/32, 3/4) \\ f_{xx} = 16, &\qquad f_{yy} = 6y \\ f_{xy} = -6 &\qquad D = 96y - 36 \end{aligned}$$

At $(0, 0), D = -36$, so saddle. At $(9/32, 3/4), D = 36, f_{xx} > 0$ Local Minimum
b)

$$\begin{aligned} \nabla f =< y^2, 2xy > &\qquad \nabla g =< 2x, 2y > \\ y^2 = 2\lambda x &\qquad 2xy = 2\lambda y, x^2 + y^2 = 4 \\ y = 0 : x = \pm 2 &\qquad (2, 0), (-2, 0) \\ y \neq 0 : x = \lambda, y^2 = 2x^2, 3x^2 = 4, &\qquad x = \pm 2/\sqrt{3}, y = \pm 2\sqrt{2}/\sqrt{3} \\ f(\pm 2, 0) = 0, f(2/\sqrt{3}, \pm 2\sqrt{2}/\sqrt{3}) = \frac{16}{3\sqrt{3}} &\qquad f(-2/\sqrt{3}, \pm 2\sqrt{2}/\sqrt{3}) = \frac{-16}{3\sqrt{3}} \end{aligned}$$