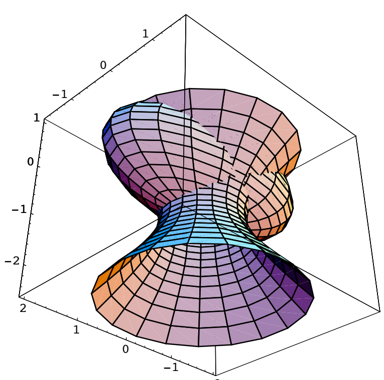
1. (10p) Find the points at which the ellipsoid $x^2/4 + y^2 + z^2 = 1$ is tangent to one of the hyperboloids in the family $x^2 + y^2 - (z+1)^2 = c^2$.

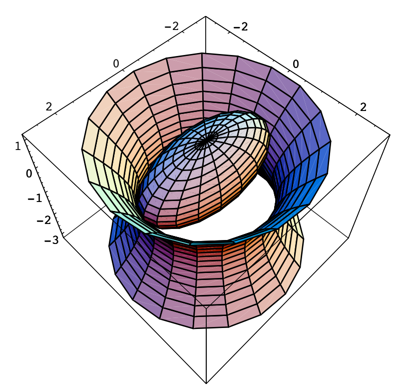
Solution We need that $\nabla(x^2/4 + y^2 + z^2) = \lambda\nabla(x^2 + y^2 - (z+1)^2)$. This means

$$\begin{aligned} x/2 &= \lambda 2x \\ 2y &= \lambda 2y \\ 2z &= -\lambda 2(z+1) \\ x^2/4 + y^2 + z^2 &= 1. \end{aligned}$$

Start with the second equation. It says $2y = \lambda 2y$, which is equivalent to $(1-\lambda)y = 0$. We have two possibilities: **a**: $\lambda = 1$ and **b**: $y = 0$.
a: If $\lambda = 1$, then from the first and third equations $x = 0$ and $z = -1/2$. Thus from the fourth we obtain that $y^2 = 3/4$ and then $y = \sqrt{3}/2$ or $y = -\sqrt{3}/2$. We found two points: $(0, \frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, -\frac{\sqrt{3}}{2}, -\frac{1}{2})$
b: If $y = 0$, then from the first equation $(2 - 8\lambda)x = 0$, and we have to subcases:
ba: $x = 0$. Then from the fourth equation $z = 1$ or $z = -1$. We found two points: $(0, 0, 1)$ and $(0, 0, -1)$.
bb: $\lambda = 1/4$. Then from the third equation $z = -1/5$ and from the fourth equation $x = \sqrt{96}/5$ or $x = -\sqrt{96}/5$. We found two points: $(\frac{\sqrt{96}}{5}, 0, -\frac{1}{5})$ and $(-\frac{\sqrt{96}}{5}, 0, -\frac{1}{5})$.

So overall we found 6 points, but we have to be careful: the points in **ba** are not good for us. $(0, 0, 1)$ is not on a hyperboloid from the family, $(0, 0, -1)$ is on the surface $x^2 + y^2 - (z+1)^2 = 0$, which is a double cone. The ellipsoid goes through the vertex of this cone. The pictures for the other four points:





2. (10p) Show that if a particle in three-dimensional space moves so that its velocity is always perpendicular to its position vector, then the particle moves on a sphere centered at the origin.

Solution Let the position of the particle be denoted by $\mathbf{c}(t)$.

$$\frac{d}{dt}||\mathbf{c}(t)||^2 = \frac{d}{dt}(\mathbf{c}(t) \cdot \mathbf{c}(t)) = 2\mathbf{c}(t) \cdot \mathbf{c}'(t) = 0,$$

which implies that the distance from the origin is constant.

3. (10p) Find the maximum and minimum value of the function

$$f(x,y) = 2x^2 + xy + \frac{5}{4}y^2 - 2x - 2y$$

on the unit square $S = [0,1] \times [0,1]$.

Solution

Interior: The gradient is $\nabla f = (4x + y - 2, x + 5y/2 - 2)$. It is 0 at $x = 1/3$, $y = 2/3$. The point $(1/3, 2/3)$ is inside the domain, so this is a candidate for abs. minimum or maximum. The value of the function is $f(1/3, 2/3) = -1$.

Boundary: It is made of four line segments.

1. $y = 0$, $0 \le x \le 1$: $f(x,y) = 2x^2 - 2$, critical point: $x = 1/2$, possible candidates for max-min: $f(0,0) = 0$, $f(1/2, 0) = -1/2$, $f(1,0) = 0$.

2. $y = 1$, $0 \le x \le 1$: $f(x,y) = 2x^2 - x - 3/4$, critical point: $x = 1/4$, possible candidates for max-min: $f(0,1) = -3/4$, $f(1/4, 1) = -1/2$, $f(1,1) = 1/4$.

3. $x = 0$, $0 \le y \le 1$: $f(x,y) = 5y^2/4 - 2y$, critical point: $y = 4/5$, possible candidates for max-min: $f(0,0) = 0$, $f(0, 4/5) = -4/5$, $f(0,1) = -3/4$.

4. $x = 1$, $0 \le y \le 1$: $f(x,y) = 5y^2/4 - y$, critical point: $y = 2/5$, possible candidates for max-min: $f(1,0) = 0$, $f(1, 2/5) = -1/5$, $f(1,1) = 1/4$.

So the absolute maximum is $f(1,1) = 1/4$, the absolute minimum is $f(1/3, 2/3) = -1$.

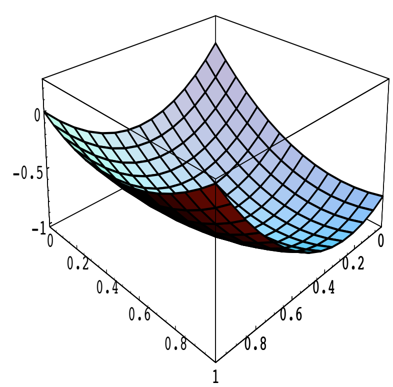


Figure 1: $f(x,y)$

4. (10p) Let $0 < a < b$ and $0 < p < q$ fixed constants. Find the area of the region which is bounded by the curves $x^2 = py$, $x^2 = qy$, $y = ax$ and $y = bx$.

Solution Change the variables by $u = x^2/y$ and $v = y/x$. Then clearly $p \leq u \leq q$ and $a \leq v \leq b$. Also,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{y}.$$

This means that the Jacobian in the change of variables formula is $y(u,v)$, which is $y = uv^2$. By this formula we obtain that the area is

$$\int_a^b \int_p^q uv^2 \, dudv = \frac{1}{6}(b^3 - a^3)(q^2 - p^2).$$

5. (10p) Find the surface area of that portion of the surface $z = y^2 + 4x$ which lies above the triangular region R in the xy-plane with vertices at $(0,0)$, $(0,2)$ and $(2,2)$.

Solution A parametrization of the surface is given by $\mathbf{\Phi}(u,v) = (u, v, v^2 + 4u)$, where the parameter domain on the uv-plane is the triangle with vertices at $(0,0)$, $(0,2)$ and $(2,2)$. Then $\mathbf{\Phi}_u \times \mathbf{\Phi}_v = (-4, -2v, 1)$ and $||\mathbf{\Phi}_u \times \mathbf{\Phi}_v|| = \sqrt{17 + 4v^2}$. The surface area we are looking for is

$$A = \int_0^2 \int_0^v \sqrt{17 + 4v^2}dudv = \int_0^2 v\sqrt{17 + 4v^2}dv =$$

$$= [\frac{1}{8}\frac{2}{3}(17 + 4v^2)^{\frac{3}{2}})]_0^2 = \frac{1}{12}(33^{\frac{3}{2}} - 17^{\frac{3}{2}}).$$

6. (10p) Evaluate the double integral

$$\int\int_R \frac{e^{y-2x}}{x+y+1} dA,$$

where R is the region bounded by $y = 2x$, $y = 2x + 3$, $y = -x$ and $y = -x + 6$.

Solution Change the variables by $u = y - 2x$ and $v = y + x$. Then clearly $0 \le u \le 3$ and $0 \le v \le 6$. Also,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3.$$

This means that the Jacobian in the change of variables formula is $1/|-3| = 1/3$. We obtain that

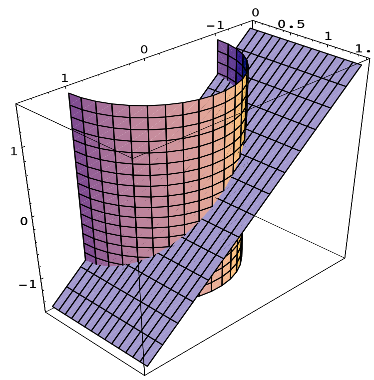
$$\int\int_R \frac{e^{y-2x}}{x+y+1} dA = \int_0^3 \int_0^6 \frac{e^u}{v+1} \frac{1}{3} dv du = \frac{1}{3}(e^3 - 1)(\ln 7).$$

7. (10p) Evaluate the line integral

$$\int_C (z^2 + 2xy)dx + x^2dy + 2xzdz$$

where C is the part of the intersection curve of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 0$ for which $y \geq 0$ and the curve is oriented in the increasing z-direction.

Solution The curve is given by the intersection of these surfaces:



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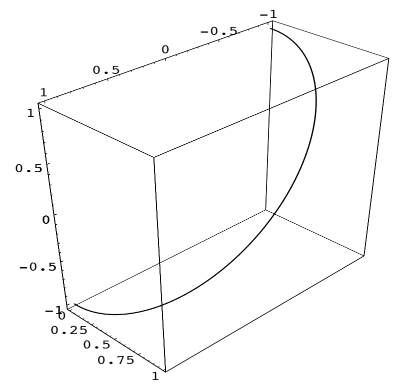

We do not have to parametrize the curve because the given vector field is a conservative field: $\nabla(z^2x+x^2y) = (z^2+2xy, x^2, 2xz)$. (We can realize this by checking that curl $\mathbf{F} = 0$.) So we need only the starting point and the endpoint of the curve, which is $(1, 0, -1)$ and $(-1, 0, 1)$, respectively. We obtain

$$\int_C (z^2 + 2xy)dx + x^2dy + 2xzdz = [z^2x + x^2y]_{(1,0,-1)}^{(-1,0,1)} = -2.$$

8. (10p) Evaluate the line integral

$$\int_C (y^3 - \ln x)dx + (\sqrt{y^2+1} + 3x)dy,$$

where C is formed by $x = y^2$ and $x = 1$ and oriented counterclockwise.

Solution By Green's theorem

$$\int_C (y^3 - \ln x)dx + (\sqrt{y^2+1} + 3x)dy = \int\int_R 3 - 3y^2 \, dA,$$

where $C = \partial R$. Thus we obtain that

$$\int_C (y^3 - \ln x)dx + (\sqrt{y^2+1} + 3x)dy = \int_{-1}^{1}\int_{y^2}^{1} 3 - 3y^2 \, dxdy = \int_{-1}^{1} (3 - 3y^2)(1 - y^2)\, dy =$$

$$= \int_{-1}^{1} 3 - 6y^2 + 3y^4 \, dy = [3y - 2y^3 + \frac{3}{5}y^5]_{-1}^{1} = \frac{16}{5}.$$

9. (10p) Let S be the unit sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{F}(x, y, z) = (zy^4 - y^2, y - x^3, z^2)$. Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

Solution By Gauss' theorem

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int\int_V \text{div } (\nabla \times \mathbf{F}) \, dV,$$

where $S = \partial V$. But div $(\nabla \times \mathbf{F}) = 0$ for any (nice enough) vector field $\mathbf{F}$, so the integral in question is 0.

10. (10p) Evaluate the flux

$$\int\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

if $\mathbf{F}(x, y, z) = (x^2 - y^2 z, 3z - \cos x, 4y^2)$ and Ω is the region bounded by $4x + 2y + z = 4$ (first octant) and the coordinate planes. The orientation is given by the outward pointing normal.

Solution By Gauss' theorem

$$\int\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{\Omega} \text{div } \mathbf{F} \, dV.$$

For the given vector field $\mathbf{F}$ div $\mathbf{F} = 2x$. The integral will become

$$\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 2x \, dz dy dx = \frac{2}{3}.$$

11. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ if C is the triangle from $(0,1,0)$ to $(0,0,4)$ to $(2,0,0)$ to $(0,1,0)$ and $\mathbf{F}(x,y,z) = (x^2 + 2xy^3z, 3x^2y^2z - y, x^2y^3)$.

Solution It is easy to check that curl $\mathbf{F} = 0$. This implies that the vector field is conservative. The given curve is closed, so the line integral is 0. (We do not even need the potential function.)

12. (10p) Verify Stokes' theorem for the vector field $\mathbf{F}(x, y, z) = (-y + x, x + y, z + z^2)$ and for the surface S which is the portion of the surface $z = 4 - x^2 - y^2$ above the xy-plane and the orientation is given by the upward pointing normal.

Solution Stokes' theorem states that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

In this case ∂S, the boundary of S is given by the intersection of the surface S and the xy-plane, so it is a circle with radius 2 in the xy-plane. Thus a parametrization is given by $\mathbf{c}(t) = (2\cos t, 2\sin t, 0)$, $0 \leq t \leq 2\pi$. By the definition of the line integral we obtain that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (-2\sin t + 2\cos t)(-2\sin t) + (2\cos t + 2\sin t)2\cos t \, dt =$$

$$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t \, dt = \int_0^{2\pi} 4 \, dt = 8\pi.$$

On the other hand, a parametrization for the surface is given by

$$\mathbf{\Phi}(u, v) = (u, v, 4 - u^2 - v^2),$$

where the parameter domain for u and v is a disk with radius 2. (Denote it by D.) We can evaluate that curl $\mathbf{F} = (0, 0, 2)$ and that $\mathbf{\Phi}_u \times \mathbf{\Phi}_v = (2u, 2v, 1)$. Thus

$$\int\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_D (0,0,2) \cdot (2u, 2v, 1) \, dudv = \int\int_D 2 \, dudv = 2A(D) = 2 \cdot 4\pi = 8\pi.$$

We verified Stokes' theorem for the given case. (The orientation of the circle was the required one for the given parametrization of the surface.)