Final Exam

1. Consider the points $P(-1,1,-1)$, $Q(-1,-2,3)$ and $R(-3,2,1)$.

a. (5 points) Find the interior angle θ of the triangle $\triangle PQR$ at vertex P.

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0,-3,4\rangle \\ \overrightarrow{PR} &= \langle -2,1,2\rangle\end{aligned} \Rightarrow \begin{aligned}\overrightarrow{PQ}\cdot\overrightarrow{PR} &= 5 \\ \left|\overrightarrow{PQ}\right| &= 5 \\ \left|\overrightarrow{PR}\right| &= 3\end{aligned} \quad \Rightarrow \theta = \boxed{\arccos(1/3)}$$

b. (5 points) Give an equation of the plane containing P, Q and R.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -10,-8,-6\rangle \Rightarrow \boxed{-10(x+1)-8(y-1)-6(z+1)=0}$$

c. (5 points) Compute the area of the triangle $\triangle PQR$.

$$A = \left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|/2 = \sqrt{200}/2 = \boxed{5\sqrt{2}}$$

2. Suppose a particle moves in space according to the formula $\mathbf{r}(t) = \left\langle t^2/2, \sqrt{2}t, \ln t \right\rangle$, where $t > 0$. Compute the following:

a. (5 points) A formula velocity, $\mathbf{v}(t)$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \boxed{\left\langle t, \sqrt{2}, 1/t \right\rangle}$$

b. (5 points) A formula for acceleration, $\mathbf{a}(t)$.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \boxed{\langle 1, 0, -1/t^2 \rangle}$$

c. (5 points) A formula for speed, $v(t)$.

$$v(t) = |\mathbf{v}(t)| = \sqrt{t^2 + 2 + 1/t^2} = \sqrt{(t + 1/t)^2} = \boxed{t + 1/t}$$

Recall that $t > 0$, by assumption.

d. (5 points) The curvature, when $t = 1$; i.e. $\kappa(1)$.

$$\begin{aligned} \mathbf{v}(1) &= \left\langle 1, \sqrt{2}, 1 \right\rangle \\ \mathbf{a}(1) &= \langle 1, 0, -1 \rangle \\ v(1) &= 2 \end{aligned} \quad \Rightarrow \kappa(1) = \frac{|\mathbf{v}(1) \times \mathbf{a}(1)|}{[v(1)]^3} = \frac{\left|\left\langle -\sqrt{2}, 2, -\sqrt{2} \right\rangle\right|}{2^3} = \boxed{\frac{\sqrt{2}}{4}}$$

e. (5 points) The tangential component of acceleration, when $t = 1$; i.e. $a_T(1)$.

$$a_T(1) = v'(1) = 1 - 1/t^2 \Big|_{t=1} = \boxed{0}$$

f. (5 points) The normal component of acceleration, when $t = 1$; i.e. $a_N(1)$.

$$a_N(1) = \kappa(1)[v(1)]^2 = \boxed{\sqrt{2}}$$

3. Let $f(x, y) = x^3 - 3x - 3y + y^3$.

a. (10 points) Determine the critical points for f.

$$\begin{aligned} f_x &= 3x^2 - 3 = 0 \\ f_y &= 3y^2 - 3 = 0 \end{aligned} \quad \Rightarrow x = \pm 1, y = \pm 1$$

So, the critical points are $\boxed{(-1, -1), (1, 1), (-1, 1) \text{ and } (1, -1)}$.

b. (10 points) Use the Second Derivative Test to classify the critical points of f.

$$D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - 0^2 = 36xy$$

Thus,

C. P.	D	f_{xx}	Type
$(-1, -1)$	36	-6	rel. max.
$(1, 1)$	36	-6	rel. min.
$(-1, 1)$	-36	n/a	saddle
$(1, -1)$	-36	n/a	saddle

(Continued from problem 3.)

c. (10 points) Use the method of Lagrange multipliers to find the maximum and minimum values of f subject to the restriction that $x^2+y^2=1$. (Hint: Repeated use of the restriction equation may help.)

First note that $x^2+y^2=1$ implies $x^2-1=-y^2$ and $y^2-1=-x^2$. Now,

$$\nabla f = \lambda \nabla g \Rightarrow \begin{array}{c} 3x^2-3=2\lambda x \\ 3y^2-3=2\lambda y \end{array} \Rightarrow \frac{3(x^2-1)}{x} = 2\lambda = \frac{3(y^2-1)}{y} \Rightarrow \frac{-y^2}{x} = \frac{-x^2}{y}$$

$$x^3=y^3 \Rightarrow x=y \Rightarrow 1=x^2+y^2=2x^2 \Rightarrow x=\pm 1/\sqrt{2}$$

(Note that we are allowed to divide by x and y above to solve for 2λ, since neither x nor y can be 0; for otherwise, our original system would imply that $-3=0$. Now, the extreme points are $(-1/\sqrt{2},-1/\sqrt{2})$ and $(1/\sqrt{2},1/\sqrt{2})$. Thus,

$$f(-1/\sqrt{2},-1/\sqrt{2}) = \boxed{5/\sqrt{2} = \text{maximum}}$$

and

$$f(1/\sqrt{2},1/\sqrt{2}) = \boxed{-5/\sqrt{2} = \text{minimum}}.$$

d. (5 points) Find the absolute maximum of f inside the unit disk centered at the origin. Remember to justify your answer.

By part a, we see that none of the critical points for f lie in the disk. So, the maximum must be achieved on the boundary; i.e. on the unit circle. Hence, by part c, the maximum of f inside the unit disk must be $\boxed{5/\sqrt{2}}$.

4. Let $f(x, y) = x^2 - y^2$.

a. (10 points) Find the unit vector that points in the direction that maximizes $D_{\mathbf{u}}f(2, -1)$.

$$\nabla f = \langle 2x, -2y \rangle \Rightarrow \nabla f(2, -1) = \langle 4, 2 \rangle \Rightarrow \mathbf{u} = \langle 4, 2 \rangle / |\langle 4, 2 \rangle| = \boxed{\langle 2/\sqrt{5}, 1/\sqrt{5} \rangle}$$

b. (10 points) Compute the maximum value of the the directional derivative of $D_{\mathbf{u}}f(2, -1)$.

$$\text{maximum} = |\nabla f(2, -1)| = \boxed{2\sqrt{5}}$$

c. (10 points) Find an equation of the tangent plane to f at $(2, -1)$.

$$z - f(2, -1) = \nabla f(2, -1) \cdot \langle x - 2, y + 1 \rangle \Leftrightarrow \boxed{z - 3 = 4(x - 2) + 2(y + 1)}$$

d. (10 points) Compute the surface area of the part of f that lies inside the cylinder $x^2 + y^2 = 4$.

$$A(S) = \iint_D \sqrt{|\nabla f|^2 + 1}dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}r dr d\theta$$

$$= 2\pi \int_1^{17} u^{1/2} \frac{du}{8} = \frac{\pi}{6} u^{3/2} \Big|_{u=1}^{17} = \boxed{\frac{\left(17\sqrt{17} - 1\right)\pi}{6}}$$

5. (15 points) Let E be the solid that lies in the first octant and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$. Suppose that the mass density of E is given by $\delta(x, y, z) = xyz$. Compute the mass of the E.

$$
\begin{aligned}
m &= \iiint_E xyz\,dV \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^{\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^{\sqrt{2}} \rho^5(\sin^3\phi\cos\phi)(\sin\theta\cos\theta)\, d\rho\, d\phi\, d\theta \\
&= \left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\cos\phi\, d\phi\right)\left(\int_1^{\sqrt{2}}\rho^5\, d\rho\right) \\
&= \left(\int_0^1 u\,du\right)\left(\int_0^1 u^3\,du\right)\left(\rho^6/6\Big|_{\rho=1}^{\sqrt{2}}\right) \\
&= \left(u^2/2\Big|_{u=0}^{1}\right)\left(u^4/4\Big|_{u=0}^{1}\right)\cdot\frac{7}{6} \\
&= \boxed{\frac{7}{48}}
\end{aligned}
$$

6. Let $\mathbf{F}(x, y) = \langle \cos(x - y), \sin y - \cos(x - y) \rangle$ be a force field on $\mathbb{R}^2$.

a. (10 points) Show that $\mathbf{F}$ is conservative by finding a potential function for it.

$$
\begin{aligned}
f_x = \cos(x - y) &\Rightarrow f(x, y) = \sin(x - y) + g(y) \\
&\Rightarrow \sin y - \cos(x - y) = f_y = -\cos(x - y) + g'(y) \\
&\Rightarrow g'(y) = \sin y \\
&\Rightarrow g(y) = -\cos y + C \\
&\Rightarrow \boxed{f(x, y) = \sin(x - y) - \cos y}
\end{aligned}
$$

Note that we are free to choose C so we pick $C = 0$.

b. (10 points) Compute the work done in moving a particle in the force field $\mathbf{F}$ along the straight line segment from the origin to the point $P(\pi/2, \pi/4)$.

$$\mathbf{W} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\pi/2, \pi/4) - f(0, 0) = \boxed{1}$$

7. (15 points) Given $P(x,y) = -y$ and $Q(x,y) = x$, verify Green's Theorem in the annulus $D = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$. (Hint: Geometry may help alleviate some of your calculations.)

Let C_R denote a the circle centered at the origin of radius R with counterclockwise (positive) orientation. Then $\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle$, for $0 \leq t \leq 2\pi$, is a parameterization of C_R, and so,

$$\int_{C_R} Pdx + Qdy = \int_0^{2\pi} (-R\sin t)(-R\sin t dt) + (R\cos t)(R\cos t dt) = \int_0^{2\pi} R^2 dt = 2\pi R^2.$$

With this notation, $\partial D = -C_1 \cup C_2$. (Note that the inner circle C_1 of the annuls has *clockwise* induced orientation.) Thus,

$$\int_{\partial D} Pdx + Qdy = -\int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy = -2\pi + 8\pi = \boxed{6\pi}.$$

On the other hand,

$$\iint_D (\partial Q/\partial x - \partial P/\partial y) dA = \iint_D 2dA = 2A(D) = 2(\pi \cdot 2^2 - \pi \cdot 1^2) = \boxed{6\pi}.$$

(We used that the area of the annulus is the area of the outer disk less the area of the inner disk.)

8. (15 points) Verify Stokes' Theorem for $\mathbf{F}(x, y, z) = \langle x, x+y, x+y+z \rangle$ over the part S of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane, where S is oriented with upward unit normal and ∂S is given the induced orientation.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & x+y & x+y+z \end{vmatrix} = \langle 1, -1, 1 \rangle$$

Now, S has parmeterization $\mathbf{r}(x, y) = \langle x, y, 1 - x^2 - y^2 \rangle$ over the disk $x^2 + y^2 \leq 1$. Since the z-component of $r_x \times r_y = \langle 2x, 2y, 1 \rangle$ is positive, we have

$$\begin{aligned}
\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle 1, -1, 1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA \\
&= \iint_D (2x - 2y + 1) dA \\
&= \int_0^{2\pi} \int_0^1 (2r\cos\theta - 2r\sin\theta + 1) r dr d\theta \\
&= \int_0^{2\pi} \left[\frac{2r^3}{3}(\cos\theta - \sin\theta) + \frac{r^2}{2} \Big|_{r=0}^{1} \right] d\theta \\
&= \int_0^{2\pi} \left[\frac{2}{3}(\cos\theta - \sin\theta) + \frac{1}{2} \right] d\theta \\
&= \frac{2}{3}(\sin\theta + \cos\theta) + \frac{\theta}{2} \Big|_{\theta=0}^{2\pi} \\
&= \boxed{\pi}.
\end{aligned}$$

The boundary of S is the circle $x^2 + y^2 = 1$, $z = 0$. It has induced parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$, for $0 \leq t \leq 2\pi$. Thus,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \cos t, \cos t + \sin t, \cos t + \sin t + 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} \cos^2 t dt \\
&= \frac{1}{2} \left(t + \frac{\sin 2t}{2} \right) \Big|_{t=0}^{2\pi} \\
&= \boxed{\pi}.
\end{aligned}$$

9. (15 points) Let E be the the solid unit cube in the first octant having the origin as one of its corners. For $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, verify the Divergence Theorem over E. (Hint: Symmetry and geometry can be used to cut down your computations. Be sure to mention when/how you use these concepts, if you decide to use them.)

Let S_1 be the face of the cube in the xy-plane. On S_1, the outward unit normal is $\langle 0, 0, -1 \rangle$ and $z = 0$. Thus,

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \langle x, y, z \rangle \cdot \langle 0, 0, -1 \rangle \, dS = \iint_{S_1} (-z) dS = 0.$$

Let S_2 be the face of the cube parallel to S_1; i.e. S_2 is just S_1 shifted 1 unit along the positive z-axis. Then on S_2, the outward unit normal is $\langle 0, 0, 1 \rangle$ and $z = 1$. Thus,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} x, y, z \cdot \langle 0, 0, 1 \rangle \, dS = \iint_{S_2} dS = A(S_2) = 1.$$

By symmetry, it follows that

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = (0 + 1) + (0 + 1) + (0 + 1) = \boxed{3}.$$

Now,

$$\iiint_E \operatorname{div} \mathbf{F} dV = \iint_E (1 + 1 + 1) dV = 3V(E) = 3 \cdot 1^3 = \boxed{3}.$$