1. A ladybug is hovering at the point $(x, y, z) = (0, 1, 2)$. The temperature is given by the function $f(x, y, z) = ze^{xy}$.

A. The temperature increases most rapidly in the direction of $\nabla f = \langle yze^{xy}, xze^{xy}, e^{xy}\rangle$. So the direction is $\langle 2, 0, 1\rangle$ with rate $|\nabla f| = \sqrt{5}$

B. What is the rate of increase in the direction $u = \langle 1, 0, 0\rangle$?
The dot product of ∇f with the unit vector u is 2.

C. The ladybug can move in any linear combination of the directions $\langle 0, 1, 0\rangle$ and $\langle \frac{1}{2}, 0, -1\rangle$ without changing temperature.

2. Suppose $f(x, y) = 2x^2 + 3y^2 - 4x - 5$.

A. Since $f_x = 4x - 4$ and $f_y = 6y$, the critical point for $f(x, y)$ is $(1, 0)$. Note that $f(1, 0) = -7$.

B. Use the method of Lagrange multipliers to find the absolute maximum and minimum of $f(x, y)$ on the region $x^2 + y^2 \leq 16$.
If $g(x, y) = x^2 + y^2$, then $\nabla f = \lambda \nabla g$ implies that $4x - 4 = \lambda 2x$ and $6y = \lambda 2y$. So $\lambda = 3$, $x = -2$ and $y = \pm 2\sqrt{3}$ or $y = 0$ and $x = \pm 4$. The maximum is $f(-2, \pm 2\sqrt{3}) = 47$ and the minimum is $f(1, 0) = -7$.

3. Evaluate the integral by reversing the order of integration: $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2} dxdy$.

This equals $\displaystyle\int_0^3 \int_0^{x/3} e^{x^2} dydx = \int_0^3 \frac{x}{3} e^{x^2} dx = (e^9 - 1)/6$.

4. A. Suppose $f(x, y) = -16 + x^2 + 4y^2$. Find the region D for which $\displaystyle\int\int_D f(x, y)dA$ is a minimum. Why is it a minimum for this region? Hint: draw a graph.
$D = \{(x, y)|x^2 + 4y^2 \leq 16\}$ because this is the region for which the function is negative.

B. Suppose $f(x, y) = 5 - x$. Let $D = \{(x, y)|0 \leq y \leq 3, 0 \leq x \leq 5\}$. Which of the following expressions is equal to $\displaystyle\int\int_D f(x, y)dA$?

iv)

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (5 - \frac{5j}{n}) \frac{5}{n} \frac{3}{m}$$

This is because $\Delta x = \frac{5}{n}$, $\Delta y = \frac{3}{m}$ and the function evaluated at the upper (or lower) right hand of each little box is $5 - x = 5 - \frac{5j}{n}$.

5. Find the volume of icecream which lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.

These intersect when $1 = 2(x^2+y^2)$ or at $r = 1/\sqrt{2}$.

$$V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} [\sqrt{1-r^2} - \sqrt{r^2}] r dr d\theta$$

With integration by substitution, this equals

$$V = 2\pi [\frac{-1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3]_0^{1/\sqrt{2}}$$

$$V = \frac{\pi}{3}(2-\sqrt{2}).$$

6. Find the surface area of the part of the plane $3x+2y+z = 6$ which lies in the first octant.

$$A(S) = \int\int_D \sqrt{1+(-3)^2+(-2)^2} dA = \sqrt{14} Area(D) = 3\sqrt{14}$$

Bonus: For positive constants a,b, and c find the surface area of the part of the plane $ax+by+z = c$ which lies in the first octant.

$$A(S) = \sqrt{1+a^2+b^2} Area(D) = \sqrt{1+a^2+b^2}\frac{c^2}{2ab}$$

7. Let x be the number of bags of candy hearts that a student can eat in an hour. Let y be the number of miles that a student can run in an hour. The joint probability density function representing the continuous random variables x and y is

$f(x,y) = \frac{1}{15}e^{-x/3}e^{-y/5}$ if $x \geq 0$ and $y \geq 0$ and $f(x,y) = 0$ otherwise.

A. Find the probability that a student can run less than one mile in an hour.

$$P = \int_0^{\infty} \frac{1}{15}e^{-x/3} dx \int_0^1 e^{-y/5} dy = 1 - e^{-1/5}.$$

B. Find the expected value (mean value) of x.

$$\mu_x = \int_0^{\infty} \frac{1}{15}e^{-y/5} dy \int_0^{\infty} x e^{-x/3} dx$$

With integration by parts,

$$\mu_x = \frac{1}{3}(-3xe^{-x/3} - 9e^{-x/3})_0^{\infty} = 3$$