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FINAL EXAM – Solutions

Problem 1. (5+5=10 points) In this problem $f(x,y) = y^2x - x^2 + 2xy$ and P is the point $P = (2,1)$.

(a) In what direction is the rate of change of f greatest at P? Express your answer in terms of a **unit** vector.

Solution. The gradient of f at the point P is the vector $\langle -1\,,\,8\rangle$. Its direction is the direction of greatest increase, so the direction expressed as a unit vector is

$$\frac{1}{|\langle -1\,,\,8\rangle|}\,\langle -1\,,\,8\rangle = \frac{1}{\sqrt{65}}\,\langle -1\,,\,8\rangle.$$

□

(b) Suppose $\vec{r}(t) = \langle x(t)\,,\,y(t)\rangle$ is a parametric curve such that $\vec{r}(0) = \langle 2\,,\,1\rangle$, and $\frac{d}{dt}\vec{r}(0) = \langle 3,\,5\rangle$. Find the value of

$$\frac{d}{dt}f(x(t)\,,\,y(t))|_{t=0}.$$

Solution. By the chain rule,

$$\frac{d}{dt}f(x(t)\,,\,y(t)) = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} = \frac{d\vec{r}}{dt}\cdot\nabla f.$$

At $t = 0$ this is the dot product $\langle 3,\,5\rangle\cdot\langle -1\,,\,8\rangle = 37$ □

Problem 2. (8+7=15 points) Suppose that, in an experiment, the temperature of a sample (in degrees Celcius) is given by the function $T(x,y,z) = 2y^2 + ze^{-x} + 16$, where x, y and z are variables one can control in the experiment.

(a) Using the linear approximation of the function T at the point $(x_0, y_0, z_0) = (0,1,2)$, find an approximate value of $T(0.2, 0.9, 2.3)$. Note that $T(0,1,2) = 20°$,

Solution. One computes that, at the point $(0,1,2)$, the partial derivatives of T are: $T_x = -2$, $T_y = 4$, $T_z = 1$. Therefore, the linearization at that point is

$$L(x,y,z) = -2(x-0) + 4(y-1) + 1(z-2) + 20.$$

Finally, one computes $L(0.2, 0.9, 2.3) = 19.5$. □

(b) Suppose one wants to change (x_0, y_0, z_0) a little and yet maintain the temperature at 20°. Using the linear approximation, **find an equation** between Δx, Δy and Δz so that

$$T(\Delta x,\,1+\Delta y,\,2+\Delta z) \cong 20.$$

Solution. By what we just said,

$$L(\Delta x,\,1+\Delta y,\,2+\Delta z) = -2\Delta x + 4\Delta y + \Delta z + 20.$$

Therefore, the equation is simply $-2\Delta x + 4\Delta y + \Delta z = 0$. □

Problem 3. ($4\times 5 = 20$ points) For each item, circle the correct answer or indicate if the statement is true or false. Assume that the functions, fields and curves below are smooth.

Think carefully before you answer –*no partial credit on this one, -take your time!*

(a) Let $\mathcal{C}$ be an arc from $(0,0)$ to $(2,1)$. According to the fundamental theorem for line integrals, $\int_{\mathcal{C}}(y-1)\,dx + (x+2y)\,dy$ is equal to:

(1) 2
(2) 1
(3) It depends on what $\mathcal{C}$ is.

Solution. The integral is equal to $\int_{\mathcal{C}} df$ where $f(x,y) = xy - x + y^2$, and therefore the fundamental theorem applies. The correct answer is $f(2,1) - f(0,0) = 1$. □

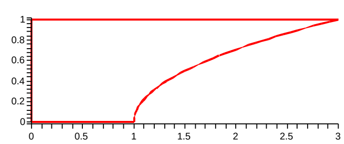
(b) For every smooth function f, the integral $\displaystyle\int_0^1 \int_0^{2y^2+1} f(x,y)\,dx\,dy$ is equal to

(1) $\int_0^3 \int_0^{\frac{1}{2}\sqrt{x-1}} f(x,y)\,dy\,dx$
(2) $\int_1^3 \int_0^{\frac{1}{2}\sqrt{x+1}} f(x,y)\,dy\,dx$
(3) None of the above.

Solution. The region of integration is plotted below. It is of type II but not of type I, because the bounds for y are:

$$0 \le y \le 1 \quad \text{for } 0 \le x \le 1, \qquad \text{and} \qquad \frac{1}{2}\sqrt{x-1} \le y \le 1 \quad \text{for } 1 \le x \le 3.$$

so the correct answer is (3). □



(c) If (a,b) is a critical point of a function f, and if

$$f_{xx}(a,b) = -2 \quad \text{and} \quad f_{yy}(a,b) = 3,$$

then what can one say about (a,b)?

(1) Nothing can be concluded from the given information.
(2) (a,b) is a local minimum of f.
(3) (a,b) is a local maximum of f.
(4) (a,b) is a saddle point of f.

Solution. It is tempting to conclude that, since we don't know anything about the value $f_{xy}(a,b)$, the correct answer should be (1). However, the discriminant of this function at (a,b) is

$$-2 \times 3 - (f_{xy}(a,b))^2 \leq -6 < 0,$$

and therefore the correct answer is (4). Another way to see this is that the function is concave down as a function of x (that is, the function $f(x,b)$ is concave down at $x=a$) and concave up as a function of y (similarly), so the point (a,b) is a saddle point of f. □

(d) If $\vec{F}$ is a field such that $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ where $\mathcal{C}$ is the unit circle, then $\vec{F}$ must be conservative.

(1) True.
(2) False.

Solution. One can't conclude that $\vec{F}$ is conservative just by knowing that the integral of $\vec{F}$ around a particular closed curve is zero. One would need to know that the integral of $\vec{F}$ around *every* closed curve is zero to conclude that $\vec{F}$ is conservative. So the correct answer is (2). □

(e) If $\mathcal{C}$ is the boundary of a domain D and $\mathcal{C}$ is oriented as in the statement of Green's theorem, then $\displaystyle\oint_{\mathcal{C}} x^2y\,dx - y\,dy$ equals

(1) $\iint_D (2xy-1)\,dA$.
(2) $\iint_D (1-x^2)\,dA$.
(3) $\iint_D (-x^2)\,dA$.
(4) None of the above.

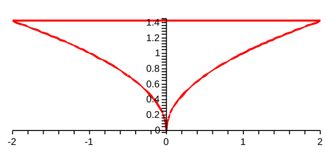
Solution. In this case $Q_x - P_y = -x^2$, so the correct answer is (3). □

Problem 4. (10 points) Let $a > 0$ denote a fixed constant. A uniform plate with mass density $2\,\mathrm{gr/cm^2}$ occupies the region bounded by the curves:

$$y = \sqrt{x} \quad \text{with } 0 \leq x \leq a, \qquad y = \sqrt{-x} \quad \text{with } -a \leq x \leq 0, \quad \text{and } y = \sqrt{a}$$

Find the coordinates of the center of mass of of the plate.

Solution. The figure below is a picture of the plate, with $a=2$. Since the plate is uniform and it is symmetric with respect to the y axis, then $\overline{x}=0$. Note that the actual value of the (constant)



density is irrelevant. (Physically, we can simply change the units of mass so that the density is

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constant equal to one without changing the location of the center of mass.) So we can take the density to be constant equal to one.

The area of the region is $A = 2\int_0^a \int_{\sqrt{x}}^{\sqrt{a}} dy\, dx = \frac{2}{3}a^{3/2}$.

Therefore

$$\overline{y} = \frac{3}{2a^{3/2}}\, 2\int_0^a \int_{\sqrt{x}}^{\sqrt{a}} y\, dy\, dx = \frac{3}{2a^{3/2}} \int_0^a (a-x)\, dx = \frac{3}{4}\sqrt{a}.$$

□

Problem 5. (5+5=10 points) Let $\mathcal{C}$ denote the oriented closed curve consisting of the line segment from $(0,0)$ to $(\sqrt{2},0)$, followed by the arc of the circle $x^2+y^2=2$ from $(\sqrt{2},0)$ to $(1,1)$, followed by the line segment from $(1,1)$ to $(0,0)$.

(a) By an explicit direct calculation, compute $I = \oint_{\mathcal{C}} y\, dx$. (You have to break the calculation into three line integrals.)

Solution. The integral over the first line segment is zero, because on the line segment $y = 0$. The integral over the arc of circle, is:

$$\int_0^{\pi/4} \sqrt{2}\sin(t)\, d(\sqrt{2}\cos(t)) = -2\int_0^{\pi/4} (\sin(t))^2\, dt = \frac{1}{2} - \frac{\pi}{4}$$

(after using a trig identity). In this calculation we used the standard parametrization $\vec{r}(t) = \langle \sqrt{2}\cos(t)\, ,\, \sqrt{2}\sin(t)\rangle$, $0 \le t \le \pi/4$.

A simple parametrization of the final segment, with the *wrong* orientation, is $\vec{r}(t) = \langle t\, ,\, t\rangle$, $0 \le t \le 1$. We can use this parametrization provided we multiply the result of the integral by (-1). In other words, the line integral over the final segment is

$$-\int_0^1 t\, dt = -1/2.$$

Therefore, the line integral around $\mathcal{C}$ is

$$\oint_{\mathcal{C}} y\, dx = \frac{1}{2} - \frac{\pi}{4} - \frac{1}{2} = -\frac{\pi}{4}.$$

□

(b) Verify your answer to part (a) by computing I using Green's theorem.

Solution. According to Green's theorem,

$$\oint_{\mathcal{C}} y\, dx = \iint_D (-1)\, dA = -\text{area of } (D),$$

where D is the domain enclosed by $\mathcal{C}$. Since this domain is one eighth of a circle of radius $\sqrt{2}$, then

$$\iint_D (-1)\, dA = -\frac{1}{8}\pi\, 2 = -\frac{\pi}{4}.$$

□

Problem 6. (7+8=15 points) Let S be the portion of the cylinder given in cylindrical cooridinates by

$$0 \le z \le 3, \qquad r = 1, \qquad 0 \le \theta \le \pi/2.$$

Orient S by normal vectors pointing away from the z axis.

(a) Compute the flux (surface integral) of $\vec{\mathrm{F}} = \langle 2x\,,\, y\,,\, -3z\rangle$ across S.

Solution. A natural parametrization of the surface is in terms of two cylindrical coordinates:

$$\vec{r}(\theta, z) = \langle \cos(\theta)\,,\, \sin(\theta)\,,\, z\rangle, \quad 0 \le \theta \le \pi/2,\ 0 \le z \le 3.$$

Note that $\vec{r}_\theta \times \vec{r}_z$ points in the direction of the orientation. A calculation shows that

$$\vec{r}_\theta \times \vec{r}_z = \cos(\theta)\vec{\imath} + \sin(\theta)\vec{\jmath},$$

and therefore the surface integral equals

$$\int_0^{\pi/2}\int_0^3 (2\cos(\theta)\vec{\imath} + \sin(\theta)\vec{\jmath} - 3z\vec{\mathrm{k}}) \cdot (\cos(\theta)\vec{\imath} + \sin(\theta)\vec{\jmath})\, dz\, d\theta$$

$$= \int_0^{\pi/2}\int_0^3 (2\cos(\theta)^2 + \sin(\theta)^2)\, dz\, d\theta = 3\int_0^{\pi/2} (\cos(\theta)^2 + 1)\, d\theta = \frac{9}{4}\pi.$$

□

(b) Let $\mathcal{C}$ denote the boundary of S, oriented counter-clockwise if one looks at S from the point $(5, 5, 1)$. Consider the line integral $I = \oint_{\mathcal{C}} yzdx - 2xzdy$.

Without computing the numerical value of I, determine whether I equals the surface integral of part (a). Justify your conclusion carefully.

Solution. By Stokes' theorem, if $\vec{G} = yz\vec{\imath} - 2xz\vec{\jmath} + 0\vec{\mathrm{k}}$, then

$$\oint_{\mathcal{C}} yzdx - 2xzdy = \iint_S \nabla \times \vec{G} \cdot d\vec{S}.$$

(Check on a picture that the orientation of $\mathcal{C}$ is such that we can apply Stoke's theorem in this form.) Now a calculation shows that

$$\nabla \times \vec{G} = \langle 2x\,,\, y\,,\, -3z\rangle = \vec{\mathrm{F}},$$

the vector field of part (a). *Therefore, by Stokes' theorem I is equal to the surface integral of part (a).* □

Problem 7. (10 points) Let E denote the portion of the solid sphere of radius R in the first octant, and let

$$\vec{\mathrm{F}} = (2x + y)\vec{\imath} + y^2\vec{\jmath} + \cos(xy)\vec{\mathrm{k}}$$

Applying the Divergence Theorem, compute the net flux of the field (surface integral) $\vec{\mathrm{F}}$ across the boundary of E, oriented by the outward-pointing normal vectors.

Solution. The divergence of $\vec{\mathrm{F}}$ is $\nabla \cdot \vec{\mathrm{F}} = 2 + 2y$. By the divergence theorem, the flux out of the given surface is equal to $\iiint_E (2 + 2y)\, dV$, where E is the region inside the surface. In spherical coordinates,

$$\iiint_E (2 + 2y)\, dV = 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (1 + \rho\sin(\theta)\sin(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$$

We split the integral as a sum of two integrals. The first one is equal to twice one eighth of the volume of the sphere of radius R, or $\frac{1}{3}\pi R^3$. The second is twice the integral of y over E, which turns out to equal

$$\frac{R^4}{2}\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta)\sin(\phi)^2\, d\theta\, d\phi = \frac{R^4}{2}\int_0^{\pi/2} \sin(\phi)^2\, d\phi = \frac{R^4}{2}\cdot\frac{\pi}{4}.$$

(using a trigonometric identity for the last step). So the final answer is

$$\frac{1}{3}\pi R^3 + \frac{1}{8}\pi R^4.$$

□

Problem 8. (5+5=10 points) The figure below is a contour plot of a function f and of its gradient. The values of f on two adjacent level curves differ by 5 units.

The plot also includes an oriented curve, $\mathcal{C}$.

(a) What is a pretty good estimate of the value of the line integral $\int_{\mathcal{C}} \nabla f \cdot d\vec{r}$?

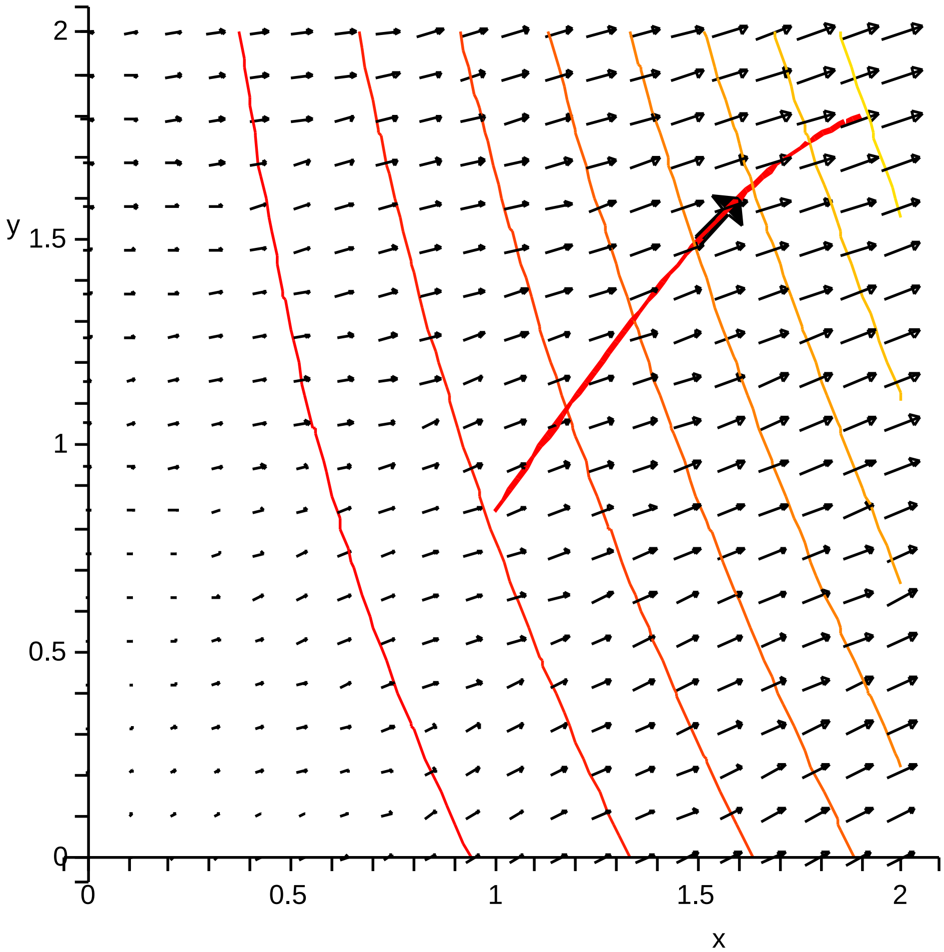
(1) 30
(2) −29
(3) The integral cannot be estimated with the given data.

Solution. We can apply the fundamental theorem for line integrals. The value of the function at the end point of $\mathcal{C}$ is approximately 30 units greater than at the initial point. This is seen by counting the level curves traversed by $\mathcal{C}$, and recalling that the gradient points in the direction of increase of the function. Therefore the correct answer is (1). □

(b) According to the plot, which of the following appears to hold?

(1) $\operatorname{div}(\nabla f)(1.7, 1) > 0$.
(2) $\operatorname{div}(\nabla f)(1.7, 1) < 0$.
(3) $\operatorname{div}(\nabla f)(1.7, 1) = 0$ because $\nabla \cdot \nabla f = 0$ for all smooth functions f.

Solution. First of all, it is *not* true that $\nabla \cdot \nabla f = 0$ for all smooth functions f. The figure shows that, in the region plotted, the magnitude of the gradient increases with x, and therefore the net flux across a tiny square is positive (more mass goes out on the right than came in at the left). Therefore the correct answer is (1). □


y
2
1.5
1
0.5
0
0
0.5
1
1.5
2
x