1. The position vector

$$\overrightarrow{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 3\sqrt{2}t\,\hat{\mathbf{j}} + 3t^{-1}\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$$

describes the motion of a particle.

(a) Find the position at time $t = 2$.

(b) Find the velocity at time $t = 2$.

(c) Find the acceleration at time $t = 2$.

(d) Find the length of the path traveled by the particle during the time $1 \le t \le 2$.

Solution:

(a) $\overrightarrow{\mathbf{r}}(2) = 8\,\hat{\mathbf{i}} + 6\sqrt{2}\,\hat{\mathbf{j}} + \frac{3}{2}\,\hat{\mathbf{k}}$

(b) The velocity is

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{r}}'(t) = 3t^2\,\hat{\mathbf{i}} + 3\sqrt{2}\,\hat{\mathbf{j}} - 3t^{-2}\,\hat{\mathbf{k}}$$

At $t = 2$ we have

$$\boxed{\overrightarrow{\mathbf{v}}(2) = 12\,\hat{\mathbf{i}} + 3\sqrt{2}\,\hat{\mathbf{j}} - \frac{3}{4}\,\hat{\mathbf{k}}}$$

(c) The acceleration is

$$\overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{r}}''(t) = 6t\,\hat{\mathbf{i}} + 6t^{-3}\,\hat{\mathbf{k}}$$

At $t = 2$ we have

$$\boxed{\overrightarrow{\mathbf{a}}(2) = 12\,\hat{\mathbf{i}} + \frac{3}{4}\,\hat{\mathbf{k}}}$$

(d) The length of the path traveled is given by the equation

$$L = \int_a^b \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right| \, dt$$

The speed $\left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|$ is given by

$$\begin{aligned}
\left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right| &= \sqrt{(3t^2)^2 + \left(3\sqrt{2}\right)^2 + (-3t^{-2})^2} \\
&= \sqrt{9t^4 + 18 + 9t^{-4}} \\
&= 3\sqrt{t^4 + 2 + t^{-4}} \\
&= 3\sqrt{(t^2 + t^{-2})^2} \\
&= 3\left(t^2 + t^{-2}\right)
\end{aligned}$$

Therefore, the length of the path traveled is

$$\begin{aligned} L &= \int_1^2 3\left(t^2 + t^{-2}\right)\, dt \\ &= 3\left[\frac{1}{3}t^3 - \frac{1}{t}\right]_1^2 \\ &= 3\left[\left(\frac{1}{3}\cdot 2^3 - \frac{1}{2}\right) - \left(\frac{1}{3}\cdot 1^3 - \frac{1}{1}\right)\right] \\ &= \boxed{\frac{17}{2}} \end{aligned}$$

2. (a) For $f(x,y) = e^{(x+1)y}$, find the derivatives:

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y^2}$$

(b) Find the gradient of f at the point $(2,3)$.

Solution:

(a) The derivatives are:

$$\frac{\partial f}{\partial x} = ye^{(x+1)y}, \quad \frac{\partial f}{\partial y} = (x+1)e^{(x+1)y}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{(x+1)y}, \quad \frac{\partial^2 f}{\partial x \partial y} = y(x+1)e^{(x+1)y}, \quad \frac{\partial^2 f}{\partial y^2} = (x+1)^2 e^{(x+1)y}$$

(b) At the point $(2,3)$ we have

$$\vec{\nabla} f(2,3) = \left\langle \frac{\partial f}{\partial x}(2,3), \frac{\partial f}{\partial y}(2,3) \right\rangle = \left\langle 3e^{(2+1)\cdot 3}, (2+1)e^{(2+1)\cdot 3} \right\rangle = \boxed{\left\langle 3e^9, 3e^9 \right\rangle}$$

3. (a) Find a potential function for the vector field

$$\vec{\mathbf{F}}(x,y,z) = (1-z)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$$

(b) Integrate $\vec{\mathbf{F}}$ over the straight line from $(1,0,1)$ to $(0,1,2)$.

Solution:

(a) One can show that a potential function is

$$\boxed{\varphi(x,y,z) = x(1-z) + \frac{1}{2}y^2}$$

To verify, we take the gradient of φ:

$$\vec{\nabla}\varphi = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = \langle 1-z, y, -x \rangle = \vec{\mathbf{F}}$$

(b) The line integral of $\overrightarrow{\mathbf{F}}$ is then

$$\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} = \varphi(0,1,2) - \varphi(1,0,1) = \left[0 \cdot (1-2) + \frac{1}{2} \cdot 1^2\right] - \left[1 \cdot (1-1) + \frac{1}{2} \cdot 0^2\right] = \boxed{\frac{1}{2}}$$

4. (a) Find the critical points of the function $f(x, y) = x^3 - 3x - y^2$.
   (b) Use the second derivative test to classify each critical point as a local maximum, a local minimum, or a saddle point.

Solution:

(a) The critical points are solutions to $f_x = 0$ and $f_y = 0$.

$$f_x = 3x^2 - 3 = 3(x-1)(x+1) = 0$$
$$f_y = -2y = 0$$

The solutions are $\boxed{(1,0),\ (-1,0)}$.

(b) The second derivatives of $f(x, y)$ are

$$f_{xx} = 6x, \qquad f_{yy} = -3, \qquad f_{xy} = 0$$

Therefore, the discriminant function is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -18x$$

The values of D at the critical points are:

$$D(1,0) = -18, \qquad D(-1,0) = 18$$

Since $D < 0$ at $(1,0)$, the point corresponds to a $\boxed{\text{saddle point}}$. Since $D > 0$ and $f_{xx} = -6 < 0$ at $(-1,0)$, the point corresponds to a $\boxed{\text{local maximum}}$.

5. Find the maximum and minimum of the function $f(x, y) = (x-1)^2 + y^2$ subject to the constraint

$$g(x,y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Solution: We will use the method of Lagrange multipliers to solve the problem. The equations $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$ and $g(x, y) = 1$ will give us the system of equations:

$$2(x-1) = \lambda\left(\frac{2}{9}x\right)$$
$$2y = \lambda\left(\frac{1}{2}y\right)$$
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

The second equation has two solutions: $y = 0$ or $\lambda = 4$. When $y = 0$, the third equation gives us $x = \pm 3$. Therefore, two critical points are:

$$(3, 0), \quad (-3, 0)$$

When $\lambda = 4$, the first equation gives us

$$2(x-1) = 4\left(\frac{2}{9}x\right) \quad \Longleftrightarrow \quad x = \frac{9}{5}$$

The third equation then gives us

$$\left(\frac{\frac{9}{5}}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \Longleftrightarrow \quad y = \pm\frac{14}{15}$$

Therefore, the other two critical points are:

$$\left(\frac{9}{5}, \frac{14}{15}\right), \quad \left(\frac{9}{5}, -\frac{14}{15}\right)$$

The values of f at the critical points are:

$$f(3,0) = 4, \quad f(-3,0) = 16, \quad f\left(\frac{9}{5}, \pm\frac{14}{15}\right) = \frac{16}{5}$$

Therefore, the minimum value of f is $\boxed{\frac{13}{5}}$ and the maximum value of f is $\boxed{16}$.

6. Compute the integral

$$\iint_{\mathcal{R}} xy\,dx\,dy$$

over the quarter circle $\mathcal{R} = \{(x, y) : 0 \le x,\, 0 \le y,\, x^2 + y^2 \le 1\}$.

Solution: Using $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$ we have:

$$\begin{aligned}
\iint_{\mathcal{R}} xy\,dx\,dy &= \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 r^3\sin\theta\cos\theta\,dr\,d\theta \\
&= \int_0^{\pi/2} \sin\theta\cos\theta\left[\frac{1}{4}r^4\right]_0^1 d\theta \\
&= \frac{1}{4}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \\
&= \frac{1}{4}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \\
&= \boxed{\frac{1}{8}}
\end{aligned}$$

7. Compute the integral

$$\iiint_{\mathcal{W}} 1\,dx\,dy\,dz$$

over the tetrahedron

$$\mathcal{W} = \{(x, y, z) \,:\, 0 \le x,\ 0 \le y,\ 0 \le z,\ x/3 + y/5 + z/7 \le 1\}$$

Solution: The value of the integral is

$$\begin{aligned}
\iiint_{\mathcal{W}} 1\,dx\,dy\,dz &= \int_0^5 \int_0^{(15-5x)/3} \int_0^{(105-21y-35x)/15} 1\,dz\,dy\,dx \\
&= \int_0^5 \int_0^{(15-5x)/3} \frac{105-21y-35x}{15}\,dy\,dx \\
&= \frac{1}{15}\int_0^5 \left[105y - \frac{21}{2}y^2 - 35xy\right]_0^{(15-5x)/3} dx \\
&= \frac{1}{15}\int_0^5 \left[105\left(\frac{15-5x}{3}\right) - \frac{21}{2}\left(\frac{15-5x}{3}\right)^2 - 35x\left(\frac{15-5x}{3}\right)\right] dx \\
&= 35\int_0^5 \left(\frac{1}{18}x^2 - \frac{1}{3}x + \frac{1}{2}\right) dx \\
&= 35\left[\frac{1}{54}x^3 - \frac{1}{6}x^2 + \frac{1}{2}x\right]_0^5 \\
&= 35\left(\frac{1}{54}\cdot 5^3 - \frac{1}{6}\cdot 5^2 + \frac{1}{2}\cdot 5\right) \\
&= \boxed{\frac{1225}{54}}
\end{aligned}$$

8. Find an equation for the tangent plane to the surface defined by $xy^2 + 2z^2 = 12$ at the point $(1, 2, 2)$.

Solution: Let $F(x, y, z) = xy^2 + 2z^2$. The gradient of F is

$$\overrightarrow{\nabla} F = \langle y^2, 2xy, 4z\rangle$$

At the point $(1, 2, 2)$ we have

$$\overrightarrow{\mathbf{n}} = \overrightarrow{\nabla} F(1, 2, 2) = \langle 4, 4, 8\rangle$$

This vector is perpendicular to the tangent plane. Using this vector and the point $(1, 2, 2)$, an equation for the tangent plane is

$$\boxed{4(x-1) + 8(y=2) + 8(z-2) = 0}$$

9. Compute the integral

$$\oint (3x^2 + y)\,dx + (x^2 + y^3)\,dy$$

over the counterclockwise boundary of the rectangle

$$\mathcal{R} = \{(x, y) \,:\, 0 \leq x \leq 3,\, 0 \leq y \leq 2\}$$

Solution: Green's Theorem is appropriate here. First, we recognize that $P = 3x^2 + y$ and $Q = x^2 + y^3$. Then,

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 1$$

The value of the integral is

$$\begin{aligned}
\oint (3x^2 + y)\, dx + (x^2 + y^3)\, dy &= \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \int_0^2 \int_0^3 (2x - 1)\; dy\, dx \\
&= \int_0^2 3(2x - 1)\, dx \\
&= \int_0^2 (6x - 3)\, dx \\
&= 3x^2 - 3x \Big|_0^2 \\
&= \boxed{6}
\end{aligned}$$