Answers

1. (a) A parametrization for the circle is $\vec{\mathbf{c}}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Then

$$\vec{\mathbf{F}}\left(\vec{\mathbf{c}}(t)\right) = \langle -\sin t, \cos t\rangle, \quad \vec{\mathbf{c}}'(t) = \langle -\sin t, \cos t\rangle$$

The value of the integral is

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_0^{2\pi} \vec{\mathbf{F}}\left(\vec{\mathbf{c}}(t)\right) \cdot \vec{\mathbf{c}}'(t)\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\, dt = 2\pi$$

(b) $\vec{\mathbf{F}}$ is not conservative. If it were, the integral in part (a) would be 0 because the path is closed.

2. (a) $\vec{\mathbf{r}}'(t) = \langle 1, e^t, 2t\rangle$

(b) $\left|\left|\vec{\mathbf{r}}'(t)\right|\right| = \sqrt{1 + e^{2t} + 4t^2}$

(c) $\vec{\mathbf{r}}''(t) = \langle 0, e^t, 2\rangle$

(d) $D = \displaystyle\int_0^3 \sqrt{1 + e^{2t} + 4t^2}\, dt$

3. (a) Let $f(x, y, z) = ze^{x^2-y^2}$. Then

$$\vec{\nabla} f = \left\langle 2xze^{x^2-y^2}, -2yze^{x^2-y^2}, e^{x^2-y^2}\right\rangle$$

A vector normal to the surface is $\vec{\mathbf{n}} = \vec{\nabla} f(1, -1, 2) = \langle 4, 4, 1\rangle$. An equation for the tangent plane at $(1, -1, 2)$ is

$$4(x - 1) + 4(y + 1) + (z - 2) = 0$$

(b) The directional derivative of f at $(1, -1, 2)$ in the direction of $\langle 2, 2, 1\rangle$ is

$$D_{\mathbf{u}} f(1, -1, 2) = \vec{\nabla} f(1, -1, 2) \cdot \hat{\mathbf{u}} = \langle 4, 4, 1\rangle \cdot \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle = \frac{17}{3}$$

4. The critical points must satisfy the equations

$$f_x = y - x = 0, \quad f_y = x + y^2 - 2 = 0$$

From the first equation we have $y = x$. Substituting this into the second equation we have $x^2 + x - 2 = 0$. The two solutions are $x = 1$ and $x = -2$. The corresponding

y-values are $y = 1$ and $y = -2$. The second derivatives are $f_{xx} = -1$, $f_{yy} = 2y$, and $f_{xy} = 1$. So,

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -2y - 1$$

At the critical points we have

$$D(1, 1) = -3 < 0 \implies \text{saddle point}$$
$$D(-2, -2) = 3 > 0, \quad f_{xx}(-2, -2) = -1 < 0 \implies \text{local maximum}$$

5. We will use the order of integration $dx\, dy$ so that

$$\iint_D e^{y^3}\, dA = \int_0^1 \int_0^{y^2} e^{y^3}\, dx\, dy = \frac{1}{3}(e - 1)$$

6. (a) $\overrightarrow{AB} = \langle 1, 2 \rangle$, $\overrightarrow{BC} = \langle -1, 1 \rangle$. The area of the parallelogram is $\left\| \overrightarrow{AB} \times \overrightarrow{BC} \right\| = 3$.

   (b) We recognize that, for the vector field $\overrightarrow{\mathbf{F}} = \langle -y, x \rangle$,

$$\oint_C -y\, dx + x\, dy = 2 \times \text{Area enclosed by } C = 2(3) = 6$$

7. Using Cylindrical Coordinates, the integral is

$$\iiint_{\mathcal{B}} x^2\, dV = \int_0^{2\pi} \int_0^1 \int_0^{r^2} (r \cos\theta)^2 r\, dz\, dr\, d\theta = \frac{\pi}{6}$$

8. (a) $\textbf{Curl}\left(\overrightarrow{\mathbf{F}}\right) = \overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \left\langle -y, 0, -x^2 \right\rangle$

   (b) $\overrightarrow{\mathbf{F}}$ is not conservative because $\textbf{Curl}\left(\overrightarrow{\mathbf{F}}\right) \neq \overrightarrow{\mathbf{0}}$

   (c) $\textbf{Div}\left(\overrightarrow{\mathbf{F}}\right) = 2xy + z + 3z^2$

   (d) $\textbf{Div}\left(\textbf{Curl}\left(\overrightarrow{\mathbf{F}}\right)\right) = 0$

9. Let $\Phi(u, v) = (u, v, 5 - 2u + 2v)$, $0 \le u \le 1$, $0 \le v \le 1$ be a parametrization of S. Then,

$$\overrightarrow{\mathbf{T}}_u = \langle 1, 0, -2 \rangle, \quad \overrightarrow{\mathbf{T}}_v = \langle 0, 1, 2 \rangle, \quad \overrightarrow{\mathbf{n}}(u, v) = \overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v = \langle 2, -2, 1 \rangle$$

The surface integral is

$$\iint_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}} = \iint_D \overrightarrow{\mathbf{F}}(\Phi(u, v)) \cdot \overrightarrow{\mathbf{n}}(u, v)\, dA = \int_0^1 \int_0^1 \langle u, v, 5 - 2u + 2v \rangle \cdot \langle 2, -2, 1 \rangle\, du\, dv$$
$$= \int_0^1 \int_0^1 (2u - 2v + 5 - 2u + 2v)\, du\, dv = 5$$