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1. Match each function to one of the given graphs, and to one of the given contour diagrams. The plots are given in a separate handout.

$f(x, y)$	Graph	Contour Diagram
$x^2 + \left(\frac{3y}{2}\right)^2$	C	8
$x^2 - \frac{1}{2}y^2$	A	4
$x + \frac{y}{2}$	B	7
$-x + \frac{y}{2}$	F	5
$\sin(\pi x)$	H	1
xy^2	E	6
$\sqrt{x^2 + \left(\frac{3y}{2}\right)^2}$	G	2
x^2	D	3

2. (a) Consider the function

$$f(x, y) = x^2 - y - xe^y - 1.$$

Find a function $g(x, y, z)$ such that the graph of f is the level surface $g(x, y, z) = 5$.

The graph of f is given by

$$z = x^2 - y - xe^y - 1,$$

which we may rewrite as $x^2 - y - xe^y - z - 1 = 0$, or, by adding 5 to both sides,

$$x^2 - y - xe^y - z + 4 = 5.$$

So we can choose

$$g(x, y, z) = x^2 - y - xe^y - z + 4.$$

(b) For each of the following functions, determine if the level surface $g(x, y, z) = 0$ can be expressed as the graph of a function $f(x, y)$. If it is not possible, explain why not. If is it possible, find the function $f(x, y)$.

i. $g(x, y, z) = x^2 + x + y^4 + z(z - 1)$

NO. If we try to solve $g(x, y, z) = 0$ for z, we first find

$$z^2 - z + x^2 + x + y^4 = 0,$$

and to solve this for z, we have to use the quadratic formula:

$$z = \frac{1}{2}\left(1 \pm \sqrt{1 - 4(x^2 + x + y^4)}\right),$$

which means there will be two z values for each point (x, y) where the expression inside the square root is not negative. So we can not express the surface as the graph of a single function $f(x, y)$.

ii. $g(x, y, z) = \sin(x - y + 2x)$

NO. The level surface $g(x, y, z) = 0$ is $\sin(x-y+2x) = 0$, which is missing the variable z. For any point (x, y) that satisfies this equation, there is a line through the point parallel to the z axis that is in the surface. So this surface can not be expressed as the graph of a function $f(x, y)$.

iii. $g(x, y, z) = 1 - e^{x^2 - y + z}$

YES. $g(x, y, z) = 0 \implies 1 - e^{x^2-y+z} = 0 \implies e^{x^2-y+z} = 1$
$\implies x^2 - y + z = \ln(1) = 0 \implies z = y - x^2$. **So the surface $g(x, y, z) = 0$ is the graph of $f(x, y) = y - x^2$.**

3. Describe the level surfaces of the function $g(x, y, z) = x^2 + 4y^2 + z$.

The level surfaces are given by $x^2 + 4y^2 + z = c$ for some constant c. We can rewrite this as $z = -x^2 - 4y^2 + c$. These surfaces are paraboloids that open downwards. (That is, they are "dome-shaped", with parabolas for cross-sections, and ellipses for contour lines.) The constant c gives the z intercept of the paraboloid.

4. Determine the points (if there are any) where the following functions are *not* continuous. Justify your answers.

   (a) $f(x, y) = \dfrac{\sin(x + y)}{x - y}$

   The numerator and denominator are both continuous, so the only points where the function would not be continuous are where the denominator is 0. So this function is not continuous along the line $y = x$. (In fact, the function is not defined there.)

   (b) $g(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

   The numerator and denominator are both continuous, and the denominator is never 0, so there are no points where this function is not continuous.

   (c) $h(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0 \\ 0 & (x, y) = (0, 0) \end{cases}$

   For $(x, y) \neq (0, 0)$, the function is a quotient of polynomials, and the denominator is not 0 if $(x, y) \neq (0, 0)$, so the function is continuous for all $(x, y) \neq (0, 0)$. To determine if the function is continuous at $(0, 0)$, we must first determine if $\lim_{(x,y)\to(0,0)} f(x, y)$ exists. Let's check a few paths towards $(0, 0)$. Along the x axis, we have $y = 0$, and

   $$\lim_{x\to 0} f(x, 0) = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

   Along the y axis, we have $x = 0$, and we find the same limit, 0. Let's try along the line $y = x$. Then

   $$\lim_{x\to 0} f(x, x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

   We see that the function approaches the value 0 along the x axis (and along the y axis), but approaches the value $1/2$ along the line $y = x$. Since the function approaches different values along two different paths to $(0, 0)$, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Therefore the function is not continuous at $(0, 0)$.

5. A train is traveling northwest at 10 miles per hour. A person in the train walks at 2 miles per hour from a window on the left side to a window directly across the train on the right side. (*Left* and *right* refer to the sides relative to a person facing the front of the train.)

Assume that $\vec{i}$ points east, and $\vec{j}$ points north. Express your answers in terms of these unit vectors.

(a) What is the velocity vector of the train?

The direction "northwest" corresponds to an angle of $3\pi/4$, so the velocity of the train is

$$10\cos(3\pi/4)\vec{i} + 10\sin(3\pi/4)\vec{j} = 10\left(-\frac{\sqrt{2}}{2}\right)\vec{i} + 10\left(\frac{\sqrt{2}}{2}\right)\vec{j} = -5\sqrt{2}\vec{i} + 5\sqrt{2}\vec{j}.$$

(b) What is the velocity vector of the person relative to the train?

Since the train is heading northwest, and the person is walking left to right across the train, the person is facing northeast. So the direction of the person relative to the train is given by an angle of $\pi/4$, and the velocity of the person relative to the train is

$$2\cos(\pi/4)\vec{i} + 2\sin(\pi/4)\vec{j} = 2\left(\frac{\sqrt{2}}{2}\right)\vec{i} + 2\left(\frac{\sqrt{2}}{2}\right)\vec{j} = \sqrt{2}\vec{i} + \sqrt{2}\vec{j}.$$

(c) What is the velocity vector of the person relative to the ground?

This is the sum of the vectors found in (a) and (b):

$$(-5\sqrt{2}\vec{i} + 5\sqrt{2}\vec{j}) + (\sqrt{2}\vec{i} + \sqrt{2}\vec{j}) = -4\sqrt{2}\vec{i} + 6\sqrt{2}\vec{j}.$$

(d) What is the speed of the person relative to the ground?

This is the magnitude of the vector found in (c):

$$\sqrt{(-4\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{104}.$$

6. (a) TRUE or FALSE? For any vectors $\vec{v}$ and $\vec{w}$, $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = \|\vec{v}\|^2 - \|\vec{w}\|^2$. (Briefly explain.)

TRUE:

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) &= \vec{v} \cdot (\vec{v} - \vec{w}) + \vec{w} \cdot (\vec{v} - \vec{w}) \\ &= (\vec{v} \cdot \vec{v}) - (\vec{v} \cdot \vec{w}) + (\vec{w} \cdot \vec{v}) - (\vec{w} \cdot \vec{w}) \\ &= (\vec{v} \cdot \vec{v}) - (\vec{w} \cdot \vec{w}) \\ &= \|\vec{v}\|^2 - \|\vec{w}\|^2.\end{aligned}$$

(Remember that $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$, and $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.)

(b) TRUE or FALSE? For any vectors $\vec{v}$ and $\vec{w}$, $\|\vec{v}+\vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.
(Briefly explain.)
FALSE. For example, take $\vec{v} = \vec{i}$ and $\vec{w} = -\vec{i}$. The $\vec{v}+\vec{w} = \vec{0}$, so $\|\vec{v}+\vec{w}\| = \|\vec{0}\| = 0$, but $\|\vec{v}\| + \|\vec{w}\| = 2$.

7. Let

$$\vec{v} = 2\vec{i} + a\vec{j} + a^2\vec{k} \qquad \text{and} \qquad \vec{w} = (2a-3)\vec{i} + \vec{j} + \vec{k}.$$

(a) What is the cosine of the angle between $\vec{v}$ and $\vec{w}$ when $a = 0$?
When $a = 0$, we have

$$\vec{v} = 2\vec{i}, \quad \textbf{and} \quad \vec{w} = -3\vec{i} + \vec{j} + \vec{k},$$

so $\|\vec{v}\| = 2$, $\|\vec{w}\| = \sqrt{11}$, and $\vec{v}\cdot\vec{w} = (2)(-3) + (0)(1) + (0)(1) = -6$. Then

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{-6}{2\sqrt{11}} = \frac{-3}{\sqrt{11}}.$$

(b) For which values of a are the vectors perpendicular? (Hint: there is at least one such value, so if you find none, check your calculation again!)
The vectors are perpendicular when $\vec{v}\cdot\vec{w} = 0$. We have

$$\vec{v}\cdot\vec{w} = (2)(2a-3) + (a)(1) + (a^2)(1) = a^2 + 5a - 6 = (a+6)(a-1).$$

So the vectors are perpendicular when $a = -6$ or $a = 1$.

8. Let

$$\vec{v} = 3\vec{i} + 2\vec{j} + \vec{k}, \quad \text{and} \quad \vec{w} = \vec{i} - \vec{j} + \vec{k}.$$

Find the following:

(a) $-2\vec{v} + \vec{w}$
$-2(3\vec{i} + 2\vec{j} + \vec{k}) + (\vec{i} - \vec{j} + \vec{k}) = (-6\vec{i} - 4\vec{j} - 2\vec{k}) + (\vec{i} - \vec{j} + \vec{k}) = \boxed{-5\vec{i} - 5\vec{j} - \vec{k}}$

(b) $\vec{v}\cdot\vec{w}$
$(3\vec{i} + 2\vec{j} + \vec{k})\cdot(\vec{i} - \vec{j} + \vec{k}) = (3)(1) + (2)(-1) + (1)(1) = \boxed{2}$

(c) A unit vector $\vec{u}$ that is parallel to $\vec{w}$.

$$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \boxed{\frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})}$$

(d) The projection of $\vec{v}$ on to $\vec{u}$, where $\vec{u}$ is the unit vector you just found in (c).

$$\vec{v}_{\text{par}} = (\vec{v}\cdot\vec{u})\vec{u} = \left(\frac{((3)(1)+(2)(-1)+(1)(1))}{\sqrt{3}}\right)\left(\tfrac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})\right) = \boxed{\frac{2}{3}(\vec{i} - \vec{j} + \vec{k})}$$

9. Given the plane

$$x + y + z = 1,$$

find the point in the plane that is closest to the point $P = (3, 3, 2)$.

Let $C = (x, y, z)$ be the point in the plane closest to P. We will find the vector $\overrightarrow{CP}$, from which we can find C.

Let $P_0 = (0, 0, 1)$; this is a point in the plane. A vector normal to the plane is $\vec{n} = \vec{i} + \vec{j} + \vec{k}$, and a unit vector in the same direction as $\vec{n}$ is

$$\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}).$$

Let $\vec{v} = \overrightarrow{P_0P} = 3\vec{i} + 3\vec{j} + \vec{k}$. Then $\overrightarrow{CP}$ is the component of $\vec{v}$ parallel to $\vec{u}$ (i.e. $\overrightarrow{CP}$ is the projection of $\vec{v}$ on $\vec{u}$). Thus

$$\overrightarrow{CP} = \vec{v}_{\mathbf{par}} = (\vec{v} \cdot \vec{u})\vec{u} = \frac{7}{3}(\vec{i} + \vec{j} + \vec{k}) = \frac{7}{3}\vec{i} + \frac{7}{3}\vec{j} + \frac{7}{3}\vec{k}.$$

Since $\overrightarrow{CP}$ is the displacement vector from C to P, we also know

$$\overrightarrow{CP} = (3 - x)\vec{i} + (3 - y)\vec{j} + (2 - z)\vec{k}.$$

Thus

$$3 - x = \frac{7}{3} \implies \boxed{x = \frac{2}{3}}, \quad 3 - y = \frac{7}{3} \implies \boxed{y = \frac{2}{3}}, \quad 2 - z = \frac{7}{3} \implies \boxed{z = -\frac{1}{3}}.$$

so $C = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.