Exam 2

Solutions

Problem 1. *(10pts) At what points (x, y) is the tangent line to the parametric curve $x(t) = t^2 + t + 1$, $y(t) = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + 1$ horizontal?*

Solution. The slope of the tangent is given by the derivative, and we know that

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{t^2 - 3t + 2}{2t + 1} = \frac{(t-1)(t-2)}{2t+1} = 0$$

if and only if $(t-1)(t-2) = 0$, that is, if $t = 1$ or $t = 2$ (note that the denominator is not zero at either of these t values). So the points at which the tangent is horizontal are

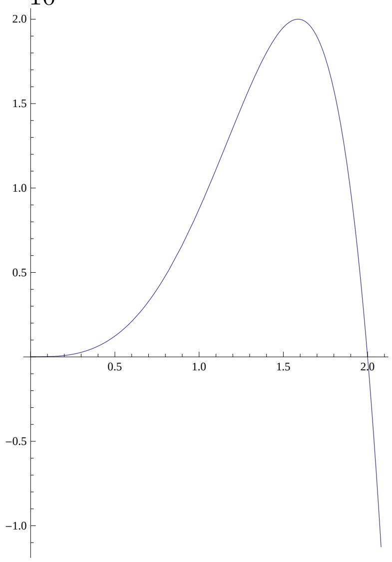
$$(x(1), y(1)) = \left(3, \frac{1}{3} - \frac{3}{2} + 2 + 1\right) = \left(3, \frac{11}{6}\right)$$

and

$$(x(2), y(2)) = \left(7, \frac{8}{3} - \frac{12}{2} + 8 + 1\right) = \left(7, \frac{17}{3}\right)$$

(but who cares about the simplification). □

Problem 2. *(10pts) Give an integral which computes the area above the x-axis and below the curve $x(t) = \sqrt[3]{t}$, $y(t) = -\frac{1}{10}\left(t^2 - 8t\right)$ for $t \geq 0$. You do not need to evaluate this integral.*



Solution. According to our formula, if the function $f(x)$ from $x = a$ to $x = b$ is traced exactly once by $x(t)$, $y(t)$, $\alpha \le t \le \beta$, then the area under the curve is

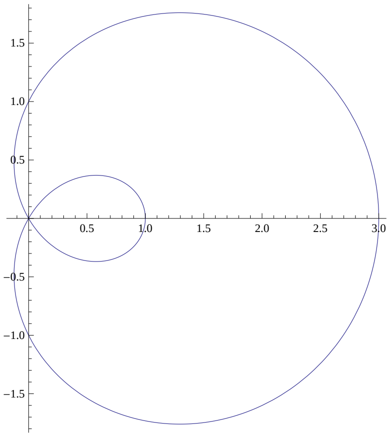
$$\int_a^b f(x)\ dx = \int_\alpha^\beta y(t)x'(t)\ dt.$$

Observe that $y = -\frac{1}{10}t(t-8) = 0$ if and only if $t = 0$ or $t = 8$, corresponding to $x(0) = 0$ and $x(8) = 2$, so we want to integrate from $x = 0$ to $x = 2$, that is, from $t = 0$ to $t = 8$. Also, $x(t)$ is an increasing function of t, so the graph is only traced once as t goes from 0 to 8. Thus the area under the curve is

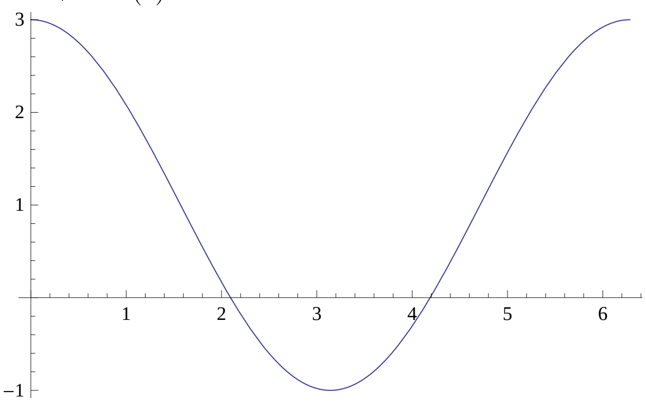
$$\int_0^8 -\frac{1}{10}\left(t^2 - 8t\right)\frac{1}{3}t^{-2/3}\ dt.$$

□

Problem 3. *(10pts) Compute the area inside inner loop of the polar curve* $r(\theta) = 1 + 2\cos(\theta)$.



Solution. We begin tracing the inner loop at the θ such that $r = 0$ first occurs, and end when $r = 0$ next occurs. The graph of $1 + 2\cos(\theta)$ is



and $r = 0$ when $1 + 2\cos(\theta) = 0$, that is, when $\cos(\theta) = -1/2$, so when $\theta = 2\pi/3, 4\pi/3$. Thus we begin tracing the inner loop at $\theta = 2\pi/3$ and finish at $\theta = 4\pi/3$. In general, the area swept by $r(\theta)$ as θ goes from α to β is $\int_\alpha^\beta \frac{1}{2}(r(\theta))^2\ d\theta$. Here area is

$$\int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2\ d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1 + 4\cos\theta + 4\cos^2\theta)$$

$$= \int_{2\pi/3}^{4\pi/3} \left(\frac{1}{2} + 2\cos\theta + 2\cos^2\theta\right)\ d\theta = \int_{2\pi/3}^{4\pi/3} \left(\frac{1}{2} + 2\cos\theta + 1 + \cos 2\theta\right)\ d\theta$$

$$= \int_{2\pi/3}^{4\pi/3} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right)\, d\theta = \left(\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{2\pi/3}^{4\pi/3}$$

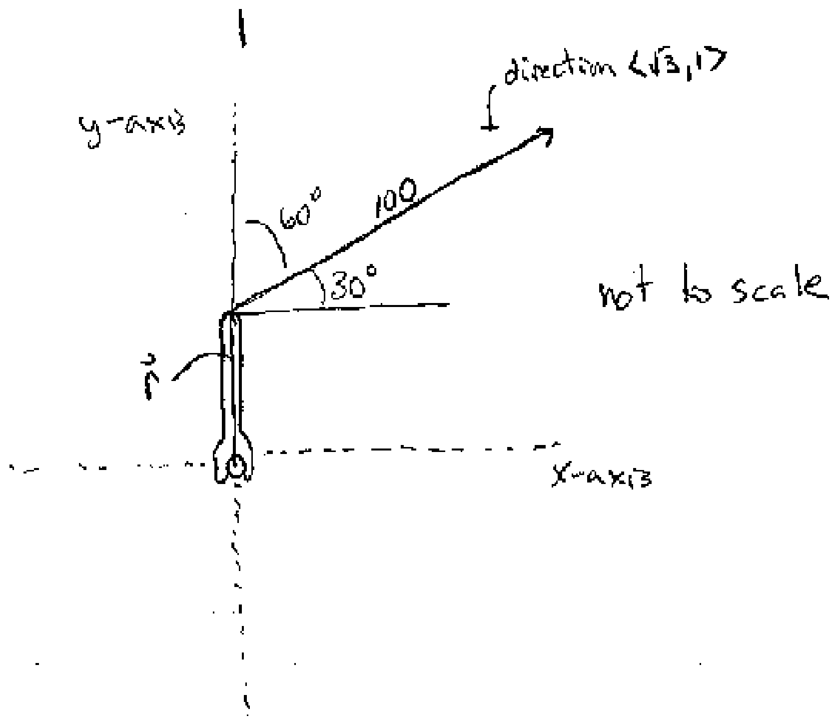
$$= \frac{3}{2}\frac{4\pi}{3} + 2\sin(4\pi/3) + \frac{\sin(8\pi/3)}{2} - \frac{3}{2}\frac{2\pi}{3} - 2\sin(2\pi/3) - \frac{\sin(4\pi/3)}{2}$$

$$= 2\pi - \sqrt{3} + \frac{\sqrt{3}}{4} - \pi - \sqrt{3} - \left(-\frac{\sqrt{3}}{4}\right) = \pi - \frac{3\sqrt{3}}{2}$$

(but who cares about the simplification; note the use of the half angle formula). □

Problem 4. *(10pts) Suppose a wrench one foot long lies along the y-axis and grips a bolt at the origin. A exceedingly large man applies a force of 100 pounds in the direction* $\langle\sqrt{3}, 1, 0\rangle$ *(which, by the way, points* 30° *degrees off the horizontal). What is the magnitude of the torque on the bolt?*

Solution. We know that torque, $\overrightarrow{\tau}$ is equal to $\overrightarrow{r} \times \overrightarrow{F}$ where $\overrightarrow{r}$ is the position vector (in this case, the vector $\langle 0, 1\rangle$) and $\overrightarrow{F}$ is the force vector (in this case, a vector of length 100 pointing in the direction $\langle\sqrt{3}, 1\rangle$). Since $\langle\sqrt{3}, 1\rangle$ lies 30° off the horizontal, and the wrench lies 90° off the horizontal, the angle between $\overrightarrow{r}$ and $\overrightarrow{F}$ must be 60° (take a look at the picture below), so according to the rule $|\overrightarrow{v} \times \overrightarrow{u}| = |\overrightarrow{v}||\overrightarrow{u}|\sin\theta$, we have $|\overrightarrow{\tau}| = |\overrightarrow{r} \times \overrightarrow{F}| = |\overrightarrow{r}||\overrightarrow{F}|\sin\theta = 1 \cdot 100 \sin 60^\circ = 100\frac{\sqrt{3}}{2} = 50\sqrt{3}$ slugs.



□

Problem 5. *(20pts) Consider the point* $(1, 2, 3)$*, the line* $\overrightarrow{r}(t) = \langle 3 + t, 2 - t, 1 + 2t\rangle$*, and the plane* $2x + y - z = 2$.

(a–10pts) Compute the distance from the point $(1, 2, 3)$ *to the plane* $2x + y - z = 2$.

Solution. Note that $(1, 0, 0)$ is a point in the plane (because it satisfies the equation $2x + y - z = 2$), and that $\overrightarrow{v} = \langle 0, 2, 3\rangle$ points from $(1, 0, 0)$ to $(1, 2, 3)$. The plane is normal to the vector $\langle 2, 1, -1\rangle$ (read of the equation of the plane). So we can compute the distance from the point to the plane by computing the length of $\langle 0,2,3\rangle$ after we project it onto the normal vector. Here

$$\text{comp}_{\langle 2,1,-1\rangle}\langle 0,2,3\rangle = \frac{\langle 2,1,-1\rangle \cdot \langle 0,2,3\rangle}{|\langle 2,1,-1\rangle||\langle 0,2,3\rangle|} = \frac{2-3}{\sqrt{2^2+1^2+(-1)^2}\sqrt{0^2+2^2+3^2}} = \frac{-1}{\sqrt{6}\sqrt{13}}.$$

Of course, we want positive distance (and our length is negative because it turns out the the normal vector we choose points away from our point instead of towards it), so we take the absolute value, and obtain the distance $\frac{1}{\sqrt{78}}$. □

(b–10pts) *What is the equation of the plane which contains the point* $(1,2,3)$ *and the line* $\overrightarrow{r}(t) = \langle 3+t, 2-t, 1+2t\rangle$*?*

Solution. To find the equation of the plane in question we need a point and a normal. We can use the point $(1,2,3)$. To find the normal, we simply take the cross product of two vectors which are parallel to the plane. One such vector is the vector pointing from $(3,2,1)$ (a point on the line and hence a point in the plane) to the point $(1,2,3)$, namely $\langle -2,0,2\rangle$. Another vector parallel to the plane is $\langle 1,-1,2\rangle$ (since this is the direction of the line and the line lies in the plane). So as a normal we take

$$\langle -2,0,2\rangle \times \langle 1,-1,2\rangle = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \langle 2,6,2\rangle,$$

and get the plane $2(x-1)+6(y-2)+2(z-3)=0$. □

Problem 6. *(8pts) Match the following graphs with their equations. You do not need to show any work for this problem.*

Equation	***Graph***
(A) $r = 1 + 2\cos(4t)$	*II*
(B) $r = 1 + 4\cos(2t)$	*IV*
(C) $x(t) = \sqrt{t} + \cos(t);\ y(t) = \sin(t)$	*III*
(D) $x(t) = t + \cos(t);\ y(t) = t$	*I*