Exam 1

Solutions

Problem 1. *(25pts) Consider the sequence* $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$.

(a–10pts) Does $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ *converge or diverge?*

Solution. In this case the sequence converges because $\lim_{n\to\infty}\frac{n}{n+1}$ exists. Using the usual nerd trick and the limit laws, we find

$$\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n}}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1.$$

□

(b–10pts) Does the sequence of partial sums of $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ *converge or diverge?*

Solution. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we use $\sum_{n=0}^{\infty} a_n$ to denote the limit of the sequence of partial sums (when it exits). So this question is asking if $\sum_{n=0}^{\infty}\frac{n}{n+1}$ converges or diverges. Since $\frac{n}{n+1} \not\to 0$ (as we saw in part (a)), we conclude that $\sum_{n=0}^{\infty}\frac{n}{n+1}$ diverges by the Test for Divergence. □

(c–5pts) Use your answer in part (b) to show that the power series $\sum_{n=0}^{\infty}\frac{n}{n+1}x^n$ *has radius of convergence* ≤ 1.

Solution. If the radius of convergence of this power series was $R > 1$, then the series would converge for all $|x| < R$, and in particular, it would converge for $x = 1$. This cannot be the case, however, since we found in part (b) that $\sum_{n=0}^{\infty}\frac{n}{n+1} = \sum_{n=0}^{\infty}\frac{n}{n+1}1^n$ diverges. It must be, therefore, that the radius of convergence is ≤ 1. □

Problem 2. *(10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.*

(a–10pts) Decide whether $\sum_{n=1}^{\infty}\frac{(-1)^n}{6^n+1}$ *converges or diverges.*

Solution. It is clear that $\sum_{n=1}^{\infty}\frac{(-1)^n}{6^n+1}$ alternates. In addition $\lim_{n\to\infty}\frac{1}{6^n+1} = 0$ (because the denominator 6^n+1 increases without bound as $n\to\infty$ and therefore $\frac{1}{6^n+1}$ becomes arbitrarily small), and $\frac{1}{6^n+1}$ is decreasing (since 6^n+1 is an increasing function). Thus $\sum_{n=1}^{\infty}\frac{(-1)^n}{6^n+1}$ satisfies the conditions for the Alternating Series Test, and hence converges. □

(b–10pts) Decide whether $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n+2}\right)^n$ *converges or diverges.*

Solution. We use the Root Test. Note that

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+1}{2n+2}\right)^n\right|}=\lim_{n\to\infty}\frac{n+1}{2n+2}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n}}\left(\frac{n+1}{2n+2}\right)=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{2}{n}}=\frac{1}{2}.$$

Since the limit is <1, we know that $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n+2}\right)^n$ converges absolutely by the Root Test. □

Problem 3. *(10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.*

(a–10pts) Decide whether $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ *converges or diverges.*

Solution. Consider the function $f(x)=\frac{1}{x\sqrt{\ln x}}$. We know that $f(x)>0$ for $x>1$, $f(x)$ is continuous for $x>1$ (being made of continuous function which are never zero for $x>1$), and $f(x)$ is decreasing for $x>1$ (since both x and $\sqrt{\ln x}$ are increasing functions (the latter being the composition of two increasing functions)). Also, we see that $f(n)=\frac{1}{n\sqrt{\ln n}}$ for all $n\in\mathbb{N}$. Thus, by the integral test, $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges if and only if $\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$ diverges.

So here we go:

$$\begin{aligned}\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx&=\lim_{b\to\infty}\int_2^{b}\frac{1}{x\sqrt{\ln x}}\,dx=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{u^{1/2}}\,dx\\&=\lim_{b\to\infty}2u^{1/2}\Big|_{\ln 2}^{\ln b}=\lim_{b\to\infty}(2\sqrt{\ln b}-2\sqrt{\ln 2})=\infty.\end{aligned}$$

We conclude that the integral (and hence also the series) diverges. Here we used the u-substitution $u=\ln x$ (whence $\frac{du}{dx}=\frac{1}{x}$, or lying, $du=\frac{dx}{x}$) and changed the limits of integration to reflect that $u=\ln 2$ when $x=2$ and $u=\ln b$ when $x=b$. We also used that fact that $\lim_{b\to\infty}\sqrt{\ln b}=\infty$.

□

(b–10pts) Decide whether $\sum_{n=1}^{\infty}\left(\frac{3n^2+1}{n^2+2}\right)\frac{1}{e^n}$ *converges or diverges.*

Solution. We compare with $\sum_{n=1}^{\infty}\frac{1}{e^n}$ using the Limit Comparison Test. Note that

$$\lim_{n\to\infty}\frac{\left(\frac{3n^2+1}{n^2+2}\right)\frac{1}{e^n}}{\frac{1}{e^n}}=\lim_{n\to\infty}\left(\frac{3n^2+1}{n^2+2}\right)=\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\left(\frac{3n^2+1}{n^2+2}\right)=\lim_{n\to\infty}\frac{3+\frac{1}{n^2}}{1+\frac{2}{n^2}}=3>0,$$

so by the Limit Comparison Test, both $\sum_{n=1}^{\infty}\left(\frac{3n^2+1}{n^2+2}\right)\frac{1}{e^n}$ and $\sum_{n=1}^{\infty}\frac{1}{e^n}$ converge or both diverge. But

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}$$

is a geometric series with $r=\frac{1}{e}<1$ and is hence convergent by the Geometric Series Test. We conclude that $\sum_{n=1}^{\infty}\left(\frac{3n^2+1}{n^2+2}\right)\frac{1}{e^n}$ converges as well. □

Problem 4. *(10pts) Compute $T_3(x)$, the Taylor polynomial of degree 3 for the function $f(x)=\sin(x)\cos(x)$.*

Solution. The formula for the Taylor polynomial of degree 3 (centered at $x=0$) is

$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=f^{(0)}(0)+f^{(1)}(0)x+\frac{f^{(2)}(0)}{2}x^2+\frac{f^{(3)}(0)}{3!}x^3.$$

So, plugging and chugging, we have:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)\cos(x)$	0
1	$-\sin^2(x)+\cos^2(x)$	1
1	$-4\sin(x)\cos(x)$	0
1	$4\sin^2(x)-4\cos^2(x)$	-4

giving the polynomial

$$T_3(x) = 0 + x + \frac{0}{2}x^2 + \frac{-4}{6}x^3 = x - \frac{2}{3}x^3.$$

□

Problem 5. *(10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.*

(a–10pts) Estimate $\int_0^{1/2} \arctan x \, dx$ *with error* $\leq \frac{1}{(9)(10)2^{10}}$. *(You may use information from your card).*

Solution. We know that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ on $(-1, 1]$ and that integrating infinite series does not change the radius of convergence. So

$$\int_0^{1/2} \arctan x \, dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} \Bigg|_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+2}}{(2n+1)(2n+2)} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+2}}{(2n+1)(2n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+1}}.$$

This series clearly alternates, while $\frac{1}{(2n+1)(2n+2)2^{2n+1}}$ is decreasing and limits to zero exactly because the denominator is an increasing function of n which grows without bound. Hence, by the Alternating Series Estimation Theorem, the error in the estimate

$$\int_0^{1/2} \arctan x \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+1}}$$

$$\approx \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+2}} = \frac{1}{(1)(2)2^2} - \frac{1}{(3)(4)2^4} + \frac{1}{(5)(6)2^6} - \frac{1}{(7)(8)2^8}$$

is at most

$$\frac{1}{(2(4)+1)(2(4)+2)2^{2(4)+2}} = \frac{1}{(9)(10)2^{10}},$$

as required.

□

(b–10pts) Estimate $e^{1/3}$ *with error* $\leq \frac{3}{5! \cdot 3^5}$. *(You may use information from your card).*

Solution. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

on $\mathbb{R}$ and that the error

$$R_k(x) = e^x - \sum_{n=0}^{k} \frac{x^n}{n!}$$

from using the degree k Taylor polynomial to estimate the function has

$$|R_k(x)| \leq \frac{M|x|^{k+1}}{(k+1)!}$$

on $|x| < R$ if $|f^{(k+1)}(x)| \leq M$ on $|x| < R$. Of course, $f^{(k+1)}(x) = e^x$ (since e^x is so fabulous), and e^x is increasing. Thus $|e^x| < e < 3$ on $|x| < 1$ and hence

$$|R_k(x)| \leq \frac{3|x|^{k+1}}{(k+1)!}$$

on $|x| < 1$. In particular,

$$|R_4(1/3)| \leq \frac{3(1/3)^5}{5!} = \frac{3}{5! \cdot 3^5},$$

so we conclude that the estimate

$$e^{1/3} \approx \sum_{n=0}^{4} \frac{(1/3)^n}{n!} = 1 + \frac{1}{3} + \frac{1}{3^2 \cdot 2!} + \frac{1}{3^3 \cdot 3!} + \frac{1}{3^4 \cdot 4!}$$

has error at most $\frac{3}{3^5 \cdot 5!}$ as required. $\square$