# First Year of Calculus

## Midterm #2 Solutions

**1.** (12 marks)

**a.** Find the Taylor polynomial of order 2 generated by the function $f(x) = x^2 + \frac{1}{x}$ at the point $x = 1$.

$$f(x) = x^2 + \frac{1}{x}$$
$$f'(x) = 2x - \frac{1}{x^2}$$
$$f''(x) = 2 + \frac{2}{x^3}$$

Therefore $f(1) = 2$, $f'(1) = 1$, and $f''(1) = 4$. Hence the order 2 Taylor polynomial is:

$$P_2(x) = 2 + 1 \bullet (x-1) + \frac{4}{2!}(x-1)^2$$
$$= 2 + (x-1) + 2(x-1)^2$$

**b.** What is the remainder $R_2(x)$ for the Taylor polynomial in part (a)? By Taylor's Theorem,

$$R_2(x) = \frac{f'''(c)}{3!}(x-1)^3$$
$$= -\frac{1}{c^4}(x-1)^3$$

where $c$ is some number between 1 and $x$.

**2.** (10 marks) Solve the initial value problem $x\frac{dy}{dx} = x - y$, $y(1) = 1$. Assume $x > 0$.

$$x\frac{dy}{dx} + y = x$$

Hence,

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 1$$

This is a linear differential equation with $P(x) = \frac{1}{x}$ and $Q(x) = 1$. Hence the solution is given by the formula

$$y(x) = \frac{1}{v(x)}\int v(x)Q(x)\,dx$$

where $v(x) = e^{\int P(x)dx}$. Since $\int P(x)\,dx = \ln|x|$, $v(x) = e^{\ln|x|} = |x|$. Since we're assuming $x > 0$, $v(x) = x$. Therefore

$$y(x) = \frac{1}{x}\int x \bullet 1\,dx$$
$$= \frac{1}{x}\left(\frac{x^2}{2} + C\right)$$
$$= \frac{x}{2} + \frac{C}{x}$$

Since $y(1) = 1$, we have $1 = \frac{1}{2} + \frac{C}{1}$ and hence $C = \frac{1}{2}$. Therefore $y(x) = \frac{x}{2} + \frac{1}{2x}$.

**3.** (10 marks) For each of the following series, state whether they converge absolutely, converge conditionally, or diverge.

**a.** This series converges absolutely: if we take absolute values of every term in the series, we get the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum \frac{1}{n^2}$. This is a p-series with $p = 2 > 1$, and hence it converges; therefore the original series converges absolutely. (Note that this is not an alternating series, so the alternating series test doesn't apply.)

**b**. This series diverges: the n$^{th}$ term of the series is $(-2)^n$, and $\lim_{n\to\infty}(-2)^n \neq 0$ (in fact, the limit does not exist). Therefore the series diverges by the n$^{th}$ term test. (Alternatively, this is a geometric series with ratio $r = -2$, and since $|r| > 1$ this geometric series must diverge.)

**4**. (10 marks) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$. Let $a_n = \frac{(2x)^n}{n}$.

Applying the ratio test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{n+1}\frac{n}{(2x)^n}\right| \\ &= \lim_{n\to\infty}\frac{n}{n+1}|2x| \\ &= 2|x|\end{aligned}$$

Hence the series converges if $2|x| < 1$, or in other words if $-\frac{1}{2} < x < \frac{1}{2}$. The center of this power series is at $x = 0$; therefore the radius of convergence of this power series is $R = \frac{1}{2}$. (The exact interval of convergence is in fact $-\frac{1}{2} \leq x < \frac{1}{2}$, but it is not necessary to show that just to compute the radius of convergence.)