First Year of Calculus

Midterm #1 Solutions

1. (10 marks) Evaluate the integral $\int_0^\infty (2x+1)^{-2}\,dx$, or state why the integral diverges.

$$\begin{aligned}\int_0^\infty (2x+1)^{-2}\,dx &= \lim_{b\to\infty}\int_0^b (2x+1)^{-2}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{2}(2x+1)^{-1}\right]_{x=0}^{b} \\ &= \lim_{b\to\infty} -\frac{1}{2}(2b+1)^{-1} - \left(-\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

2. (10 marks) Evaluate the limits of the following sequences. Justify your answers.

a.

$$\lim_{n\to\infty}\frac{n+(-1)^n}{n} = \lim_{n\to\infty}\frac{1+\frac{(-1)^n}{n}}{1}$$

Since $\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$, by the Sandwich Theorem $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$. Therefore $\lim_{n\to\infty}\frac{n+(-1)^n}{n} = \frac{1+0}{1} = 1$.

b. Using L'Hopital's rule,

$$\begin{aligned}\lim_{n\to\infty}\frac{n+\ln(n)}{n^2+\ln(n)} &= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2n+\frac{1}{n}} \\ &= \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{n^2}}{2+\frac{1}{n^2}} \\ &= \frac{0+0}{2+0} = 0\end{aligned}$$

3. (10 marks) Evaluate the following infinite series, or state why they diverge:

a. $-\frac{4}{3}+\frac{4}{9}-\frac{4}{27}+\frac{4}{81}\cdots+(-1)^n\frac{4}{3^n}+\cdots$ is a geometric series, with $a=-\frac{4}{3}$ and $r=-\frac{1}{3}$. Since $|r|<1$, the series converges to $\frac{a}{1-r} = \frac{-4/3}{1-(-1/3)} = -1$.

b. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ is a telescoping series: we have

$$\begin{aligned}s_1 &= 1-\frac{1}{\sqrt{2}} \\ s_2 &= \left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) = 1-\frac{1}{\sqrt{3}} \\ s_3 &= \left(1-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right) = 1-\frac{1}{\sqrt{4}}\end{aligned}$$

and in general, $s_n = 1-\frac{1}{\sqrt{n+1}}$. Therefore $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} 1-\frac{1}{\sqrt{n+1}} = 1$.

4. (10 marks) Determine whether or not the following series converge or diverge:

a. $\sum_{n=1}^{\infty}\frac{n+1}{2n+3}$ diverges according to the nth-term test, since $\lim_{n\to\infty}\frac{n+1}{2n+3} = \frac{1}{2} \neq 0$.

b. $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ converges according to the Ratio test, since

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \bullet \frac{(2n)!}{2^n} \\ &= \lim_{n\to\infty} \frac{2 \bullet 2^n}{(2n+2)(2n+1)(2n)!} \bullet \frac{(2n)!}{2^n} \\ &= \lim_{n\to\infty} \frac{2}{(2n+2)(2n+1)} \\ &= 0\end{aligned}$$

Hence $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ and the series converges.