CHAPTER SIX

CONTINGENT ANNUITY MODELS

6-1 Start with the identity derived in Example 6.2 and simply multiply both sides by $(1+i)$, obtaining

$$(1+i)A_x = 1-i \cdot a_x,$$

and the desired relationship follows.

6-2 We begin with Equation (5.30), where $X = A_x$. Then we can write

$$\begin{aligned}
A_{x+1} \cdot \ell_{x+1} &= v \cdot d_{x+1} + v^2 \cdot d_{x+2} + v^3 \cdot d_{x+3} + \cdots, \\
A_{x+2} \cdot \ell_{x+2} &= v \cdot d_{x+2} + v^2 \cdot d_{x+3} + v^3 \cdot d_{x+4} + \cdots, \\
A_{x+3} \cdot \ell_{x+3} &= v \cdot d_{x+3} + v^2 \cdot d_{x+4} + v^3 \cdot d_{x+5} + \cdots,
\end{aligned}$$

and so on to the end of the survival model where $\ell_\omega = 0$. Summing this set of equations we have

$$\begin{aligned}
\sum_{t=1}^{\infty} A_{x+t} \cdot \ell_{x+t} &= v(d_{x+1} + d_{x+2} + \cdots) + v^2(d_{x+2} + d_{x+3} + \cdots) \\
&\quad + v^3(d_{x+3} + d_{x+4} + \cdots) + \cdots \\
&= v \cdot \ell_{x+1} + v^2 \cdot \ell_{x+2} + v^3 \cdot \ell_{x+3} + \cdots \\
&= a_x \cdot \ell_x, \text{ by Equation (6.3).}
\end{aligned}$$

6-3 From Equation (6.4b) we have

$$\begin{aligned} a_x &= v \cdot p_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \cdots \\ &= v \cdot p_x (1 + v \cdot p_{x+1} + v^2 \cdot {}_2p_{x+1} + \cdots) \\ &= v \cdot p_x (1 + a_{x+1}). \end{aligned}$$

6-4 From Equation (6.15) we have

$$Var(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{d^2}.$$

We are given values of $\ddot{a}_x$ and ${}^2\ddot{a}_x$, so we use the relationship derived in Example 6.4 to write

$$Var(\ddot{Y}_x) = \frac{(1 - d' \cdot {}^2\ddot{a}_x) - (1 - d \cdot \ddot{a}_x)^2}{d^2}.$$

Here $i = \frac{1}{24}$ so $d = \frac{i}{1+i} = \frac{1/24}{25/24} = \frac{1}{25} = .04$. Furthermore, $i' = (1+i)^2 - 1 = \left(\frac{25}{24}\right)^2 - 1 = .08507$, so $d' = \frac{.08507}{1.08507} = .07840$. Then we have

$$\begin{aligned} Var(\ddot{Y}_x) &= \frac{[1 - (.07840)(6)] - [1 - (.04)(10)]^2}{(.04)^2} \\ &= \frac{.52960 - .36000}{.0016} = 106. \end{aligned}$$

6-5. Begin with the relationship found in Example 6.4 and substitute $d = 1 - v$ to obtain

$$A_x = 1 - (1-v) \cdot \ddot{a}_x = v \cdot \ddot{a}_x - (\ddot{a}_x - 1) = v \cdot \ddot{a}_x - a_x,$$

from Equation (6.10).

6-6 Recall that

$$a_x = v\cdot p_x + v^2\cdot {}_2p_x + \cdots$$
$$= v\cdot p_x(1+v\cdot p_{x+1}+v^2\cdot {}_2p_{x+1}+\cdots) = v\cdot p_x\cdot \ddot{a}_{x+1}.$$

Then from Equation (6.10) we have

$$\ddot{a}_x = 1+a_x = 1+v\cdot p_x\cdot \ddot{a}_{x+1},$$

as required.

6-7 From Exercise 6-6 we have

$$\ddot{a}_x = 1+v\cdot p_x\cdot \ddot{a}_{x+1}$$
$$= 1+v\cdot p_x\left(1+v\cdot p_{x+1}+v^2\cdot {}_2p_{x+1}+\cdots\right)$$
$$= 1+v\cdot p_x\left[1+v\cdot p_{x+1}(1+v\cdot p_{x+2}+\cdots)\right].$$

In the initial calculation $p_x = .99$ and $p_{x+1} = .95$, so we have

$$\ddot{a}_x = 1+\frac{.99}{1.05}\left[1+\frac{.95}{1.05}\cdot \ddot{a}_{x+2}\right].$$

But also we know that $\ddot{a}_{x+1} = 6.951$, so we can find

$$\ddot{a}_{x+2} = \frac{(6.951-1)(1.05)}{.95} = 6.57742.$$

Now using $p'_{x+1} = p_{x+1}+.03 = .98$ we find

$$\ddot{a}'_x = 1+\frac{.99}{1.05}\left[1+\frac{.98}{1.05}\cdot \ddot{a}_{x+2}\right],$$

so the increase is

$$\ddot{a}'_x-\ddot{a}_x = \frac{(.99)(.98)}{(1.05)^2}\cdot \ddot{a}_{x+2}-\frac{(.99)(.95)}{(1.05)^2}\cdot \ddot{a}_{x+2}$$
$$= \frac{6.57742}{(1.05)^2}\left((.99)(.98)-(.99)(.95)\right) = .17719.$$

6-8 If the death benefit is B, then the present value random variable for the new product is

$$\begin{aligned} W_x &= 12{,}000\ddot{Y}_x + B \cdot Z_x \\ &= 12{,}000\left(\frac{1-Z_x}{d}\right) + B \cdot Z_x \\ &= \frac{12{,}000}{.08} + Z_x\left(B - \frac{12{,}000}{.08}\right) \\ &= 150{,}000 + (B - 150{,}000) \cdot Z_x. \end{aligned}$$

Then

$$Var(W_x) = (B - 150{,}000)^2 \cdot Var(Z_x),$$

which is minimized at $B = 150{,}000$.

6-9 We write Equation (6.14) as

$$\begin{aligned} E[\ddot{Y}_x] &= \sum_{k=1}^{\infty}\left(\frac{1-v^k}{d}\right) \cdot {}_{k-1|}q_x \\ &= \frac{1}{d}\left[\sum_{k=1}^{\infty} {}_{k-1|}q_x - \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_x\right] \\ &= \frac{1}{d}[1 - A_x] \\ &= \ddot{a}_x, \end{aligned}$$

using the result of Example 6.4.

6-10 This is the continuous counterpart of the result of Example 6.4. Rearranging Equation (6.18) we have

$$\bar{Z}_x = 1 - \delta \cdot \bar{Y}_x,$$

and taking the expectation yields the desired relationship.

6-11 We use the definition of $\bar{a}_x$ given by Equation (6.17b), so we have

$$\frac{d}{dx}\bar{a}_x = \frac{d}{dx}\int_0^\infty v^t \cdot {}_tp_x\,dt$$
$$= \int_0^\infty v^t \cdot \left(\frac{d}{dx}\,{}_tp_x\right)dt$$
$$= \int_0^\infty v^t \cdot \left[{}_tp_x(\mu_x - \mu_{x+t})\right]dt,$$

using the result of Exercise 4-6(b). Then we have

$$\frac{d}{dx}\bar{a}_x = \mu_x\int_0^\infty v^t \cdot {}_tp_x\,dt - \int_0^\infty v^t \cdot {}_tp_x\mu_{x+t}\,dt$$
$$= \mu_x \cdot \bar{a}_x - \bar{A}_x$$
$$= \mu_x \cdot \bar{a}_x - (1-\delta \cdot \bar{a}_x)$$
$$= \bar{a}_x(\mu_x+\delta) - 1,$$

as required.

6-12 Recall that $E[\bar{Y}_x] = \bar{a}_x$. We use the result of Exercise 6-10 to write

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}.$$

Then we use the result of Equation (5.49a), where $\bar{A}_x$ was derived under the uniform distribution, to write

$$E[\bar{Y}_x] = \bar{a}_x = \frac{1}{\delta}\left[1 - \frac{\bar{a}_{\overline{\omega-x}|}}{\omega - x}\right] = \frac{(\omega-x) - \bar{a}_{\overline{\omega-x}|}}{\delta(\omega-x)},$$

as required.

6-13 Recall that

$$Var(\bar{a}_{\overline{T_x}|}) = Var(\bar{Y}_x) = \frac{{}^2\bar{A}_x - \bar{A}_x^{\,2}}{\delta^2}.$$

We know that

$$\bar{A}_x = 1-\delta\cdot\bar{a}_x = 1-10\delta$$

and

$${}^2\bar{A}_x = 1-2\delta\cdot{}^2\bar{a}_x = 1-14.75\delta.$$

Then

$$Var(\bar{Y}_x) = \frac{(1-14.75\delta)-(1-10\delta)^2}{\delta^2} = \frac{5.25\delta-100\delta^2}{\delta^2} = 50,$$

which solves for $\delta = .035$. Finally

$$\bar{A}_x = 1-(.035)(10) = .65.$$

6-14 Since μ_{x+t} is constant, then T_x has an exponential distribution. From Example 6.5 we know that, in general, $\bar{a}_x = \frac{1}{\mu+\delta}$. Here we have $\mu = k$ and $\delta = 4k$, so

$$\begin{aligned} Var(\bar{a}_{\overline{T_x}|}) &= \frac{{}^2\bar{A}_x - \bar{A}_x^{\,2}}{\delta^2} \\ &= \frac{(1-2\delta\cdot{}^2\bar{a}_x)-(1-\delta\cdot\bar{a}_x)^2}{\delta^2} \\ &= \frac{\left[1-8k\left(\frac{1}{k+8k}\right)\right]-\left[1-4k\left(\frac{1}{k+4k}\right)\right]^2}{(4k)^2} \\ &= \frac{\left(1-\frac{8}{9}\right)-\left(1-\frac{4}{5}\right)^2}{16k^2} = \frac{100}{9}, \end{aligned}$$

which solves for $k = .02$.

6-15 In general,

$$\begin{aligned} Pr(\bar{Y}_x > 20) &= Pr\left(\frac{1-v^T}{\delta} > 20\right) \\ &= Pr\left(1-e^{-\delta T} > 20\delta\right) \\ &= Pr\left(e^{-\delta T} < 1-20\delta\right) \\ &= Pr\left[-\delta T < \ln(1-20\delta)\right] \\ &= Pr\left[T > \frac{\ln(1-20\delta)}{-\delta}\right]. \end{aligned}$$

Here $\delta = .03$ so we have

$$Pr(T > 30.54302).$$

But T is exponential with parameter $\mu = .025$, so

$$Pr(T > 30.54302) = e^{-(.025)(30.54302)} = .46600.$$

6-16 We have

$$\begin{aligned} Cov(\bar{Y}_x, \bar{Z}_x) &= E[\bar{Y}_x \cdot \bar{Z}_x] - E[\bar{Y}_x] \cdot E[\bar{Z}_x] \\ &= E\left[\frac{1-\bar{Z}_x}{\delta} \cdot \bar{Z}_x\right] - E[\bar{Y}_x] \cdot E[\bar{Z}_x] \\ &= \frac{1}{\delta} \cdot E\left[\bar{Z}_x - \bar{Z}_x^{\,2}\right] - E[\bar{Y}_x] \cdot E[\bar{Z}_x] \\ &= \frac{1}{\delta}\left[\bar{A}_x - {}^2\bar{A}_x\right] - \bar{a}_x \cdot \bar{A}_x \\ &= \frac{1}{\delta}\left[\bar{A}_x - {}^2\bar{A}_x\right] - \frac{1}{\delta}(1-\bar{A}_x) \cdot \bar{A}_x \\ &= \frac{\bar{A}_x}{\delta} - \frac{{}^2\bar{A}_x}{\delta} - \frac{\bar{A}_x}{\delta} + \frac{\bar{A}_x^{\,2}}{\delta} \\ &= \frac{\bar{A}_x^{\,2} - {}^2\bar{A}_x}{\delta}, \end{aligned}$$

as required.

6-17 Recall that ${}_tp_x = e^{-\int_0^t \mu_{x+r}\,dr}$, so we can write $\bar{a}_x$ as

$$\bar{a}_x = \int_0^\infty v^t \cdot {}_tp_x\,dt = \int_0^\infty e^{-\delta t} \cdot e^{-\int_0^t \mu_{x+r}\,dr}\,dt.$$

Similarly,

$$\bar{a}_x^{\,*} = \int_0^\infty e^{-\delta^* t} \cdot e^{-\int_0^t \mu_{x+r}^*\,dr}\,dt$$

$$= \int_0^\infty e^{-3\delta t} \cdot e^{-\int_0^t \mu_{x+r}^*\,dr}\,dt.$$

But $\bar{a}_x = \bar{a}_x^{\,*}$, so we have

$$e^{-\delta t} \cdot e^{-\int_0^t \mu_{x+r}\,dr} = e^{-3\delta t} \cdot e^{-\int_0^t \mu_{x+r}^*\,dr}$$

$$= e^{-\delta t} \cdot e^{-2\delta t} \cdot e^{-\int_0^t \mu_{x+r}^*\,dr}$$

$$= e^{-\delta t} \cdot e^{-\int_0^t (\mu_{x+r}^* + 2\delta)\,dr},$$

which shows that $\mu_{x+r} = \mu_{x+r}^* + 2\delta$, as required.

6-18 Recall that $Var(\bar{Y}_x) = \dfrac{{}^2\bar{A}_x - \bar{A}_x^{\,2}}{\delta^2}$ and $E[\bar{Z}_x] = \bar{A}_x$. The unconditional first moment of $\bar{Z}_x$ is

$$(.50)\left(E[\bar{Z}_x^{\,M}] + E[\bar{Z}_x^{\,F}]\right) = (.50)(.09+.15) = .1200.$$

We are given the values of $Var(\bar{Y}_x)$ for males and females separately, so we can find

$${}^2\bar{A}_x^{\,M} = (5.00)(.10)^2 + (.15)^2 = .0725$$

and

$${}^2\bar{A}_x^{\,F} = (4.00)(.10)^2 + (.09)^2 = .0481.$$

The unconditional second moment of $\bar{Z}_x$ is then

$$(.50)\left({}^2\bar{A}_x^{\,M} + {}^2\bar{A}_x^{\,F}\right) = (.50)(.0725 + .0481) = .0603.$$

Then the unconditional variance of $\bar{Y}_x$ is

$$\frac{.0603 - (.1200)^2}{(.10)^2} = 4.59.$$

6-19 Taking the expectation of Equation (6.22b) we have

$$E\left[Y_{x:\overline{n}|}\right] = a_{x:\overline{n}|} = \frac{1}{i}\left[1 - E[U] - E[V]\right]$$
$$= \frac{1}{i}\left[1-(1+i)\cdot A^{1}_{x:\overline{n}|} - A_{x:\overset{\;\;1}{\overline{n}|}}\right],$$

using Equations (5.7) and (5.20), respectively, to find $E[U]$ and $E[V]$. Then we have

$$i \cdot a_{x:\overline{n}|} = 1 - i \cdot A^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|} - A_{x:\overset{\;\;1}{\overline{n}|}} = 1 - i \cdot A^{1}_{x:\overline{n}|} - A_{x:\overline{n}|},$$

and the desired relationship follows.

6-20 As in Section 6.1.1, if X is paid by each of ℓ_x persons then the initial fund is $X \cdot \ell_x$ which we equate to the present value of all annuity payments. Then we have

$$X \cdot \ell_x = v \cdot \ell_{x+1} + v^2 \cdot \ell_{x+2} + \cdots + v^n \cdot \ell_{x+n}.$$

Dividing by ℓ_x produces

$$X = v \cdot \frac{\ell_{x+1}}{\ell_x} + v^2 \cdot \frac{\ell_{x+2}}{\ell_x} + \cdots + v^n \cdot \frac{\ell_{x+n}}{\ell_x}$$
$$= v \cdot p_x + v^2 \cdot {}_2p_x + \cdots + v^n \cdot {}_np_x,$$

which is $a_{x:\overline{n}|}$ by Equation (6.21).

6-21 From Equation (6.23) we have

$$
\begin{aligned}
E\left[Y_{x:\overline{n}|}\right] &= \sum_{k=1}^{n}\left(\frac{1-v^{k-1}}{i}\right)\cdot {}_{k-1|}q_x + a_{\overline{n}|}\sum_{k=n+1}^{\infty} {}_{k-1|}q_x \\
&= \frac{1}{i}\left[\sum_{k=1}^{n} {}_{k-1|}q_x - \sum_{k=1}^{n} v^{k-1}({}_{k-1}p_x - {}_kp_x)\right] + a_{\overline{n}|}\cdot {}_np_x \\
&= \frac{1}{i}\left[1 - {}_np_x + (1-v^n)\cdot {}_np_x - \sum_{k=1}^{n} v^{k-1}({}_{k-1}p_x - {}_kp_x)\right] \\
&= \frac{1}{i}\left[1 - v^n\cdot {}_np_x - \sum_{k=1}^{n} v^{k-1}\cdot {}_{k-1}p_x + \sum_{k=1}^{n}(1+i)\cdot v^k\cdot {}_kp_x\right] \\
&= \frac{1}{i}\left[(1+i)\sum_{k=1}^{n} v^k\cdot {}_kp_x - \left(\sum_{k=1}^{n} v^{k-1}\cdot {}_{k-1}p_x - 1 + v^n\cdot {}_np_x\right)\right] \\
&= \frac{1}{i}\left[(1+i)\sum_{k=1}^{n} v^k\cdot {}_kp_x - \sum_{k=1}^{n} v^k\cdot {}_kp_x\right] \\
&= \sum_{k=1}^{n} v^k\cdot {}_kp_x,
\end{aligned}
$$

which is $a_{x:\overline{n}|}$ by Equation (6.21).

6-22 This time the present value of all annuity payments includes ℓ_x payments at age x but no payments at age $x+n$. We have

$$X\cdot \ell_x = \ell_x + v\cdot \ell_{x+1} + \cdots + v^{n-1}\ell_{x+n-1}.$$

Dividing by ℓ_x produces

$$X = 1 + v\cdot p_x + \cdots + v^{n-1}\cdot {}_{n-1}p_x,$$

which is $\ddot{a}_{x:\overline{n}|}$ by Equation (6.27).

6-23 This is similar to Exercise 6-21. We have

$$
\begin{aligned}
E\left[\ddot{Y}_{x:\overline{n}|}\right] &= \sum_{k=1}^{n}\left(\frac{1-v^k}{d}\right)\cdot {}_{k-1|}q_x + \ddot{a}_{\overline{n}|}\sum_{k=n+1}^{\infty} {}_{k-1|}q_x \\
&= \frac{1}{d}\left[\sum_{k=1}^{n} {}_{k-1|}q_x - \sum_{k=1}^{n} v^k\left({}_{k-1}p_x - {}_kp_x\right)\right] + \ddot{a}_{\overline{n}|}\cdot {}_np_x \\
&= \frac{1}{d}\left[1 - {}_np_x + (1-v^n)\cdot {}_np_x - \sum_{k=1}^{n} v^k\left({}_{k-1}p_x - {}_kp_x\right)\right] \\
&= \frac{1}{d}\left[1 - v^n\cdot {}_np_x + \sum_{k=1}^{n} v^k\cdot {}_kp_x - \sum_{k=1}^{n} v^k\cdot {}_{k-1}p_x\right] \\
&= \frac{1}{d}\left[\sum_{k=0}^{n-1} v^k\cdot {}_kp_x - v\sum_{k=1}^{n} v^{k-1}\cdot {}_{k-1}p_x\right] \\
&= \frac{1}{d}\left[\sum_{k=0}^{n-1} v^k\cdot {}_kp_x - (1-d)\sum_{k=0}^{n-1} v^k\cdot {}_kp_x\right] = \sum_{k=0}^{n-1} v^k\cdot {}_kp_x,
\end{aligned}
$$

which is $\ddot{a}_{x:\overline{n}|}$ by Equation (6.27).

6-24 Beginning with Equation (6.24b), we first divide numerator and denominator by $(1+i)^2$ to obtain

$$
Var(Y_{x:\overline{n}|}) = \frac{\left({}^2A^1_{x:\overline{n}|} - \left(A^1_{x:\overline{n}|}\right)^2\right) + v^2\left({}^2A_{x:\overline{n}|}^{\;\;\,1} - \left(A_{x:\overline{n}|}^{\;\;\,1}\right)^2\right) - 2v\cdot A^1_{x:\overline{n}|}\cdot A_{x:\overline{n}|}^{\;\;\,1}}{d^2}.
$$

Then we rearrange terms in the numerator to obtain

$$
\begin{aligned}
Var(Y_{x:\overline{n}|}) &= \frac{\left({}^2A^1_{x:\overline{n}|} + v^2\cdot {}^2A_{x:\overline{n}|}^{\;\;\,1}\right) - \left(\left(A^1_{x:\overline{n}|}\right)^2 + 2v\cdot A^1_{x:\overline{n}|}\cdot A_{x:\overline{n}|}^{\;\;\,1} + v^2\cdot \left(A_{x:\overline{n}|}^{\;\;\,1}\right)^2\right)}{d^2} \\
&= \frac{\left({}^2A^1_{x:\overline{n}|} + v^2\cdot {}^2A_{x:\overline{n}|}^{\;\;\,1}\right) - \left(A^1_{x:\overline{n}|} + v\cdot A_{x:\overline{n}|}^{\;\;\,1}\right)^2}{d^2}.
\end{aligned}
$$

Observe that the expression inside the parentheses of the squared term in the numerator can be written as

$$A^{1}_{x:\overline{n}|} + v \cdot A_{x:\overline{n}|}^{\ \ \,1} = v \cdot q_x + \cdots + v^n \cdot {}_{n-1|}q_x + v \cdot v^n \cdot {}_n p_x .$$

We add and subtract the term $v^{n+1} \cdot {}_{n+1}p_x$ to this expression, obtaining

$$v \cdot q_x + \cdots + v^n \cdot {}_{n-1|}q_x + v^{n+1} \cdot {}_n p_x - v^{n+1} \cdot {}_{n+1}p_x + v^{n+1} \cdot {}_{n+1}p_x$$

$$= v \cdot q_x + \cdots + v^n \cdot {}_{n-1|}q_x + v^{n+1} \cdot {}_{n|}q_x + {}_{n+1}E_x$$

$$= A^{1}_{x:\overline{n+1}|} + A_{x:\overline{n+1}|}^{\ \ \ \ \ \,1} = A_{x:\overline{n+1}|} .$$

The expression inside the first set of parentheses in the numerator is the same as that in the second set, except at interest rate $i' = (1+i)^2 - 1$ (or $\delta' = 2\delta$). Therefore the same steps as above will produce ${}^2A_{x:\overline{n+1}|}$ for this expression so we have

$$Var(Y_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n+1}|} - A_{x:\overline{n+1}|}^{\ 2}}{d^2},$$

which is Equation (6.34) with n replaced by $n+1$.

6-25 The process is the same as that shown on page 178, except here we have payment of ℓ_x at age x but no payment at age $x+n$. Then we have

$$X = \frac{\ell_x(1+i)^n + \ell_{x+1}(1+i)^{n-1} + \cdots + \ell_{x+n-1}(1+i)}{\ell_{x+n}}$$

$$= \frac{(1+i)^n \cdot \ell_x}{\ell_{x+n}} \left[\frac{\ell_x(1+i)^n + \ell_{x+1}(1+i)^{n-1} + \cdots + \ell_{x+n-1}(1+i)}{(1+i)^n \cdot \ell_x} \right]$$

$$= \frac{(1+i)^n \cdot \ell_x}{\ell_{x+n}} \left[\frac{\ell_x + v \cdot \ell_{x+1} + \cdots + v^{n-1} \cdot \ell_{x+n-1}}{\ell_x} \right]$$

$$= \frac{(1+i)^n \cdot \ell_x}{\ell_{x+n}} \left[1 + v \cdot p_x + \cdots + v^{n-1} \cdot {}_{n-1}p_x \right]$$

$$= \frac{(1+i)^n \cdot \ell_x}{\ell_{x+n}} \cdot \ddot{a}_{x:\overline{n}|},$$

which is Equation (6.35).

6-26 From Equation (6.31) we have

$$\begin{aligned}\ddot{a}_{x:\overline{4}|} &= E\left[\ddot{Y}_{x:\overline{4}|}\right] \\ &= \ddot{a}_{\overline{1}|}\cdot q_x + \ddot{a}_{\overline{2}|}\cdot {}_{1|}q_x + \ddot{a}_{\overline{3}|}\cdot {}_{2|}q_x + \ddot{a}_{\overline{4}|}\cdot {}_{3|}q_x + \ddot{a}_{\overline{4}|}\cdot {}_4p_x.\end{aligned}$$

Note that $\ddot{a}_{\overline{4}|}$ is the present value of payments if (x) fails in the fourth year or survives beyond the fourth year, so $\ddot{a}_{\overline{4}|}$ is the value if (x) survives beyond the third year, the probability of which is ${}_3p_x$. Then we have

$$\begin{aligned}\ddot{a}_{x:\overline{4}|} &= (1.00)(.33)+(1.93)(.24) \\ &\quad +(2.80)(.16)+(3.62)(1-.33-.24-.16) \\ &= .3300+.4632+.4480+.9774 \\ &= 2.2186.\end{aligned}$$

6-27 We have

$$\begin{aligned}E\left[\ddot{Y}_{x:\overline{3}|}\right] &= (1)(1-p_x)+(1.87)(p_x-{}_2p_x)+(2.62)({}_2p_x) \\ &= (1)(.10)+(1.87)(.90-.81)+(2.62)(.81) \\ &= .1000+.1683+2.1222 = 2.3905\end{aligned}$$

and

$$\begin{aligned}E\left[\ddot{Y}_{x:\overline{3}|}^{\,2}\right] &= (1)^2(.10)+(1.87)^2(.90-.81)+(2.62)^2(.81) \\ &= .10000+.31472+5.56016 = 5.97488.\end{aligned}$$

Then

$$Var\left(\ddot{Y}_{x:\overline{3}|}\right) = 5.97488-(2.3905)^2 = .26039.$$

6-28 From Equation (6.40) we have

$$
\begin{aligned}
Var\left(\bar{Y}_{x:\overline{n}|}\right) &= \frac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}_{x:\overline{n}|}^{\,2}}{\delta^2} \\
&= \frac{\left(1-2\delta\cdot{}^2\bar{a}_{x:\overline{n}|}\right)-\left(1-\delta\cdot\bar{a}_{x:\overline{n}|}\right)^2}{\delta^2} \\
&= \frac{1-2\delta\cdot{}^2\bar{a}_{x:\overline{n}|}-1+2\delta\cdot\bar{a}_{x:\overline{n}|}-\delta^2\cdot\bar{a}_{x:\overline{n}|}^{\,2}}{\delta^2} \\
&= \frac{1}{\delta}\left(-2\cdot{}^2\bar{a}_{x:\overline{n}|}+2\cdot\bar{a}_{x:\overline{n}|}-\delta\cdot\bar{a}_{x:\overline{n}|}^{\,2}\right) \\
&= \frac{2}{\delta}\left(\bar{a}_{x:\overline{n}|}-{}^2\bar{a}_{x:\overline{n}|}\right)-\bar{a}_{x:\overline{n}|}^{\,2},
\end{aligned}
$$

as required.

6-29 We have

$$
\begin{aligned}
\int_0^n \bar{a}_{\overline{t}|}\cdot{}_tp_x\mu_{x+t}\,dt &= \frac{1}{\delta}\int_0^n (1-v^t)\cdot{}_tp_x\mu_{x+t}\,dt \\
&= \frac{1}{\delta}\left({}_nq_x-\bar{A}^{\,1}_{x:\overline{n}|}\right) \\
&= \frac{1}{\delta}\left[(1-{}_np_x)-(1-\delta\cdot\bar{a}_{x:\overline{n}|}-{}_nE_x)\right] \\
&= \frac{1}{\delta}\left(\delta\cdot\bar{a}_{x:\overline{n}|}-{}_np_x(1-v^n)\right) \\
&= \bar{a}_{x:\overline{n}|}-{}_np_x\cdot\bar{a}_{\overline{n}|},
\end{aligned}
$$

as required.

6-30 From Equation (6.47) we have

$$
\begin{aligned}
E[{}_{n|}Y_x] &= \sum_{k=n+1}^{\infty} v^n\left(\frac{1-v^{k-n-1}}{i}\right){}_{k-1|}q_x \\
&= \frac{1}{i}\left[v^n \sum_{k=n+1}^{\infty} {}_{k-1|}q_x - \sum_{k=n+1}^{\infty} v^{k-1}\cdot {}_{k-1|}q_x\right] \\
&= \frac{1}{i}\left[v^n\cdot {}_np_x - \sum_{k=n+1}^{\infty} v^{k-1}({}_{k-1}p_x - {}_kp_x)\right] \\
&= \frac{1}{i}\left[-\left(\sum_{k=n+1}^{\infty} v^{k-1}\cdot {}_{k-1}p_x - v^n\cdot {}_np_x\right) + \sum_{k=n+1}^{\infty} v^{k-1}\cdot {}_kp_x\right] \\
&= \frac{1}{i}\left[(1+i)\sum_{k=n+1}^{\infty} v^k\cdot {}_kp_x - \sum_{k=n+1}^{\infty} v^k\cdot {}_kp_x\right] \\
&= \sum_{k=n+1}^{\infty} v^k\cdot {}_kp_x = \sum_{s=1}^{\infty} v^{s+n}\cdot {}_{s+n}p_x,
\end{aligned}
$$

which is ${}_{n|}a_x$ by Equation (6.45).

6-31 This demonstration parallels that of Example 6.9. In this case,

$$ {}_{n|}\ddot{Y}_x = \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|}, $$

so

$$ {}_{n|}\ddot{Y}_x = \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{K_x+1}|} = 0 $$

for $K_x \le n$, and

$$ {}_{n|}\ddot{Y}_x = \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|} = v^n\cdot \ddot{a}_{\overline{K_x+1-n}|} $$

for $K_x > n$, as required. Then

$$ E\left[{}_{n|}\ddot{Y}_x\right] = E\left[\ddot{Y}_x - \ddot{Y}_{x:\overline{n}|}\right] = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = {}_nE_x\cdot \ddot{a}_{x+n} = {}_{n|}\ddot{a}_x, $$

as required.

6-32 The force of mortality is constant, so ${}_tp_x = e^{-\mu t}$ for all t. Then we can calculate

$$\begin{aligned}\ddot{a}_x &= 1 + v \cdot p_x + v^2 \cdot {}_2p_x + \cdots \\ &= 1 + e^{-\delta} \cdot e^{-\mu} + e^{-2\delta} \cdot e^{-2\mu} + \cdots \\ &= 1 + e^{-(\mu+\delta)} + (e^{-(\mu+\delta)})^2 + \cdots \\ &= \frac{1}{1-e^{-(\mu+\delta)}} = \frac{1}{1-e^{-[.01+\ln(1.04)]}} = 20.82075,\end{aligned}$$

so

$$_{5|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{5|}} = 16.27875.$$

Then $S > {}_{5|}\ddot{a}_x$ if 17 payments are made, which occurs if (x) survives to age $x+21$. This means that

$$Pr(S > {}_{5|}\ddot{a}_x) = {}_{21}p_x = e^{-(21)(.01)} = .81058.$$

6-33 The death benefit is a 30-year term insurance of the net single premium amount, so we have the equation value

$$NSP = NSP \cdot A^1_{35:\overline{30|}} + {}_{30|}\ddot{a}_{35} = NSP \cdot A^1_{35:\overline{30|}} + {}_{30}E_{35} \cdot \ddot{a}_{65}.$$

We find ${}_{30}E_{35}$ as

$${}_{30}E_{35} = A_{35:\overline{30|}}^{\;\;\,1} = A_{35:\overline{30|}} - A^1_{35:\overline{30|}} = .14,$$

so

$$NSP = \frac{(.14)(9.90)}{1-.07} = 1.49032.$$

6-34 From the definition of $\bar{Y}$ we have

$$E[\bar{Y}] = \int_0^n \bar{a}_{\overline{n}|} \cdot {}_tp_x\mu_{x+t}\,dt + \int_n^\infty \bar{a}_{\overline{t}|} \cdot {}_tp_x\mu_{x+t}\,dt$$

$$= \int_0^\infty \bar{a}_{\overline{t}|} \cdot {}_tp_x\mu_{x+t}\,dt - \int_0^n \bar{a}_{\overline{t}|} \cdot {}_tp_x\mu_{x+t}\,dt + \int_0^n \bar{a}_{\overline{n}|} \cdot {}_tp_x\mu_{x+t}\,dt$$

$$= \bar{a}_x - \left(\bar{a}_{x:\overline{n}|} - {}_np_x \cdot \bar{a}_{\overline{n}|}\right) + \bar{a}_{\overline{n}|} \cdot {}_nq_x,$$

where we use the result of Exercise 6-29. This simplifies to

$$\bar{a}_x - \bar{a}_{x:\overline{n}|} + {}_np_x \cdot \bar{a}_{\overline{n}|} + \bar{a}_{\overline{n}|} - {}_np_x \cdot \bar{a}_{\overline{n}|} = {}_{n|}\bar{a}_x + \bar{a}_{\overline{n}|},$$

as required. The result is also intuitive. Payment is made for n years in any case, whether (x) survives or not, and then beyond $t = n$ if survival occurs. Thus the APV is $\bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$.

6-35 (a) Equation (6.66): Start with Equation (6.61), and imitate the variable change that evolved Equation (6.44) into Equation (6.45).

(b) Equation (6.67): Start with Equation (6.64), and imitate the variable change that evolved Equation (6.48b) into Equation (6.51).

(c) Equation (6.68): Simply add the right sides of Equations (6.59) and (6.61) to obtain the right side of Equation (6.58).

(d) Equation (6.69): Simply add the right sides of Equations (6.63) and (6.64) to obtain the right side of Equation (6.62)

(e) Equation (6.70): Simply add $\frac{1}{m}$ to the right side of Equation (6.58), by including $t = 0$ in the summation, to obtain the right side of Equation (6.62).

6-36 (a) On the right side of Equation (6.59), add $\frac{1}{m}$ (by including $t=0$ in the summation) and subtract $\frac{1}{m} \cdot {}_nE_x$ (by deleting $t=mn$ in the summation) to reach the right side of Equation (6.63).

(b) On the right side of Equation (6.61), add $\frac{1}{m} \cdot {}_nE_x$ (by including $t=mn$ in the summation) to reach the right side of Equation (6.64).

6-37 Begin with the definition of $\ddot{Y}_x^{(m)}$ given by Equation (6.76). We have

$$\ddot{Y}_x^{(m)} = \frac{1}{m}\left(\frac{1-v^{J_x}}{d}\right),$$

where this d is the effective discount rate over $\frac{1}{m}$ of a year. Then $m \cdot d = d^{(m)}$, so we have

$$\ddot{Y}_x^{(m)} = \frac{1-v^{J_x}}{d^{(m)}} = \frac{1-Z_x^{(m)}}{d^{(m)}},$$

where $Z_x^{(m)}$ is defined by Equation (5.39). Taking expectations yields

$$\ddot{a}_x^{(m)} = E\left[\ddot{Y}_x^{(m)}\right] = \frac{1-E[Z_x^{(m)}]}{d^{(m)}} = \frac{1-A_x^{(m)}}{d^{(m)}},$$

and Equation (6.77) follows.

6-38 Let $\alpha(m) = \dfrac{id}{i^{(m)}d^{(m)}}$ and $\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)}d^{(m)}}$.

(a) From Equation (6.67),

$$
\begin{aligned}
{}_{n|}\ddot{a}_x^{(m)} &= {}_nE_x \cdot \ddot{a}_{x+n}^{(m)} \\
&= {}_nE_x\left[\alpha(m)\cdot \ddot{a}_{x+n} - \beta(m)\right] \\
&= \alpha(m)\cdot {}_{n|}\ddot{a}_x - \beta(m)\cdot {}_nE_x .
\end{aligned}
$$

(b) From Equation (6.69),

$$
\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} &= \ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)} \\
&= \left(\alpha(m)\cdot \ddot{a}_x - \beta(m)\right) - \left(\alpha(m)\cdot {}_{n|}\ddot{a}_x - \beta(m)\cdot {}_nE_x\right) \\
&= \alpha(m)\cdot(\ddot{a}_x - {}_{n|}\ddot{a}_x) - \beta(m)\cdot(1 - {}_nE_x) \\
&= \alpha(m)\cdot \ddot{a}_{x:\overline{n}|} - \beta(m)\cdot(1 - {}_nE_x).
\end{aligned}
$$

(c) From Equation (6.65),

$$
\begin{aligned}
\ddot{s}_{x:\overline{n}|}^{(m)} &= \frac{1}{{}_nE_x}\cdot \ddot{a}_{x:\overline{n}|}^{(m)} \\
&= \frac{1}{{}_nE_x}\left(\alpha(m)\cdot \ddot{a}_{x:\overline{n}|} - \beta(m)\cdot(1 - {}_nE_x)\right) \\
&= \alpha(m)\cdot \ddot{s}_{x:\overline{n}|} - \beta(m)\cdot\left(\frac{1}{{}_nE_x} - 1\right).
\end{aligned}
$$

6-39 (a) Begin with Equation (6.70), rewritten as

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m},$$

and substitute for $\ddot{a}_x^{(m)}$ from Equation (6.78). We have

$$a_x^{(m)} = \alpha(m) \cdot \ddot{a}_x - \beta(m) - \frac{1}{m}.$$

Then substitute $\ddot{a}_x = a_x + 1$, reaching

$$\begin{aligned} a_x^{(m)} &= \alpha(m) \cdot a_x + \alpha(m) - \beta(m) - \frac{1}{m} \\ &= \frac{id}{i^{(m)} d^{(m)}} \cdot a_x + \left(\frac{id}{i^{(m)} d^{(m)}} - \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} - \frac{1}{m} \right) \\ &= \frac{id}{i^{(m)} d^{(m)}} \cdot a_x + \frac{m(i-d) - m(i-i^{(m)}) - m(i^{(m)} - d^{(m)})}{m \cdot i^{(m)} d^{(m)}} \\ &= \frac{id}{i^{(m)} d^{(m)}} \cdot a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}}, \end{aligned}$$

as required. Then with part (a) established, each of parts (b), (c), and (d) follow from part (a) in the same way Exercise 6-38, parts (a), (b), and (c) followed from Equation (6.78).

6-40 (a) The result follows immediately from

$$\delta = \lim_{m \to \infty} i^{(m)} = \lim_{m \to \infty} d^{(m)}.$$

(b) The result follows immediately for the same reason.

(c) Starting with the part (b) result we have

$$\begin{aligned} \bar{a}_x = \frac{id}{\delta^2} (\ddot{a}_x - 1) + \frac{\delta - d}{\delta^2} &= \frac{id}{\delta^2} \cdot \ddot{a}_x - \frac{id}{\delta^2} + \frac{\delta - d}{\delta^2} \\ &= \frac{id}{\delta^2} \cdot \ddot{a}_x - \left(\frac{id - \delta + d}{\delta^2} \right) \\ &= \frac{id}{\delta^2} \cdot \ddot{a}_x - \left(\frac{i - d - \delta + d}{\delta^2} \right) \\ &= \frac{id}{\delta^2} \cdot \ddot{a}_x - \frac{i - \delta}{\delta^2}, \end{aligned}$$

which is the part (a) result.

6-41 (a) $${}_{n|}\ddot{a}_x^{(m)} = {}_nE_x \cdot \ddot{a}_{x+n}^{(m)}$$
$$\approx {}_nE_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right) = {}_{n|}\ddot{a}_x - \frac{m-1}{2m}\cdot {}_nE_x$$

(b) $$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)}$$
$$\approx \left(\ddot{a}_x - \frac{m-1}{2m}\right) - \left({}_{n|}\ddot{a}_x - \frac{m-1}{2m}\cdot {}_nE_x\right)$$
$$= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

(c) $$\ddot{s}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} \cdot \frac{1}{{}_nE_x} \approx \left[\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)\right]\cdot\frac{1}{{}_nE_x}$$
$$= \ddot{s}_{x:\overline{n}|} - \frac{m-1}{2m}\left(\frac{1}{{}_nE_x} - 1\right)$$

(d) $$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$$
$$\approx \ddot{a}_x - \frac{m-1}{2m} - \frac{1}{m}$$
$$= a_x + 1 - \frac{m-1}{2m} - \frac{1}{m}$$
$$= a_x + \frac{2m - m + 1 - 2}{2m} = a_x + \frac{m-1}{2m}$$

(e) $${}_{n|}a_x^{(m)} = {}_nE_x \cdot a_{x+n}^{(m)} \approx {}_nE_x\left(a_{x+n} + \frac{m-1}{2m}\right)$$
$$= {}_{n|}a_x + \frac{m-1}{2m}\cdot {}_nE_x$$

(f) $$a_{x:\overline{n}|}^{(m)} = a_x^{(m)} - {}_{n|}a_x^{(m)} \approx \left(a_x + \frac{m-1}{2m}\right) - \left({}_{n|}a_x + \frac{m-1}{2m}\cdot {}_nE_x\right)$$
$$= a_{x:\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x)$$

(g) $$s_{x:\overline{n}|}^{(m)} = a_{x:\overline{n}|}^{(m)} \cdot \frac{1}{{}_nE_x} \approx \left[a_{x:\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x)\right]\cdot\frac{1}{{}_nE_x}$$
$$= s_{x:\overline{n}|} + \frac{m-1}{2m}\left(\frac{1}{{}_nE_x} - 1\right)$$

6-42 By L'Hospital's Rule,

$$\lim_{m\to\infty} \frac{m-1}{2m} = \lim_{m\to\infty} \frac{1}{2} = \frac{1}{2},$$

so $\ddot{a}_x^{(m)}$ becomes

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2}$$

and $a_x^{(m)}$ becomes

$$\bar{a}_x \approx a_x + \frac{1}{2}.$$

6-43 Recall that $\bar{A}_x = \frac{i}{\delta} \cdot A_x$, and $A_x = 1 - d \cdot \ddot{a}_x$, so we have

$$\begin{aligned}\bar{A}_x &= \frac{i}{\delta}(1 - d \cdot \ddot{a}_x) \\ &= \frac{i}{\delta} - \frac{id}{\delta} \cdot \ddot{a}_x \\ &= \frac{i}{\delta} - \frac{i-d}{\delta} \cdot \ddot{a}_x.\end{aligned}$$

6-44 The APV of the contract is

$$\begin{aligned}APV &= 1 + 2 \cdot v \cdot p_x + 3 \cdot v^2 \cdot {}_2p_x \\ &= 1 + (2)(.90)(.80) + (3)(.90)^2(.80)(.75) \\ &= 3.898.\end{aligned}$$

The present value of payments actually made is $1 + 2v = 2.80$ if only two payments are made, and $2.80 + 3v^2 = 5.23$ if all three payments are made. Then for the present value of payments actually made to exceed the APV, survival to time $t = 2$ is required, the probability of which is $(.80)(.75) = .60$.

6-45 The present value random variable Y is $y=2$ for failure in the first year (with probability .20), $y=2+3v=4.70$ for failure in the second year (with probability $(.80)(.25)=.20$), and $y=4.70+4v^2=7.94$ for survival to the third year (with probability $(.80)(.75)=.60$). Then we have

$$E[Y] = (2)(.20)+(4.70)(.20)+(7.94)(.60) = 6.104$$

and

$$E[Y^2] = (2)^2(.20)+(4.70)^2(.20)+(7.94)^2(.60) = 43.04416,$$

so $Var(Y) = (43.04416)-(6.104)^2 = 5.78534.$

6-46 (a) Extending the result of Exercise 5-25(a), we have

$$\begin{aligned}(IA)_x &= A_x + {}_1E_x \cdot A_{x+1} + {}_2E_x \cdot A_{x+2} + \cdots \\ &= (1-d\cdot\ddot{a}_x) + {}_1E_x(1-d\cdot\ddot{a}_{x+1}) + {}_2E_x(1-d\cdot\ddot{a}_{x+2}) + \cdots \\ &= (1+{}_1E_x+{}_2E_x+\cdots) \\ &\qquad -d(\ddot{a}_x + {}_1E_x\cdot\ddot{a}_{x+1} + {}_2E_x\cdot\ddot{a}_{x+2}+\cdots) \\ &= \ddot{a}_x - d\cdot(I\ddot{a})_x,\end{aligned}$$

as required.

(b) From Equation (5.55) we have

$$(\bar{I}\bar{A})_x = \int_0^\infty t\cdot v^t \cdot {}_tp_x\mu_{x+t}\,dt.$$

Using integration by parts we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \frac{t\cdot v^t}{-\delta\cdot t\cdot v^t+v^t}\left|\frac{{}_tp_x\mu_{x+t}\,dt}{-{}_tp_x}\right.$$

$$= -t\cdot v^t\cdot {}_tp_x\Big|_0^\infty - \int_0^\infty(\delta\cdot t\cdot v^t - v^t)\cdot {}_tp_x\,dt.$$

The first term goes to zero at both limits, so we have

$$\begin{aligned}(\bar{I}\bar{A})_x &= \int_0^\infty v^t\cdot{}_tp_x\,dt - \delta\int_0^\infty t\cdot v^t\cdot{}_tp_x\,dt \\ &= \bar{a}_x - \delta\cdot(\bar{I}\bar{a})_x,\end{aligned}$$ as required.