since $Var(Z_x)$ is a constant with respect to B. Equating to zero we easily find $B = 150{,}000$. (Note that this value of B minimizes the variance at $Var(X) = 0$, making the combined contract risk-free to the insurer.)

6.7 EXERCISES

6.1 Whole Life Annuity Models

6-1 Derive the identity $i \cdot a_x + (1+i)A_x = 1$.

6-2 Use a group deterministic interpretation to show that

$$\sum_{t=1}^{\infty} \ell_{x+t} \cdot A_{x+t} = \ell_x \cdot a_x.$$

6-3 Show that $a_x = v \cdot p_x (1+a_{x+1})$. (Note that when $i = 0$ this equation reduces to that shown in Exercise 4-18.)

6-4 Calculate the value of $Var(\ddot{Y}_x)$, given the values $\ddot{a}_x = 10$, ${}^2\ddot{a}_x = 6$, and $i = \frac{1}{24}$.

6-5 Show that $A_x = v \cdot \ddot{a}_x - a_x$.

6-6 Show that $\ddot{a}_x = 1 + v \cdot p_x \cdot \ddot{a}_{x+1}$.

6-7 After calculating the value of $\ddot{a}_x$ at interest rate $i = .05$, a student discovers that the value of p_{x+1} is larger by .03 than the value used in the initial calculation. Find the amount by which the value of $\ddot{a}_x$ is increased when the correct value of p_{x+1} is used, given the following values used in the initial calculation:

$$q_x = .01 \qquad q_{x+1} = .05 \qquad \ddot{a}_{x+1} = 6.951$$

6-8 A whole life annuity product pays the contract holder 12,000 at the beginning of each year. It is suggested that a death benefit, payable at the end of the year of death, be added to the product. Find the size of the death benefit that will minimize the variance of the present value random variable of the new product, given $d=.08$.

6-9 Show that the right side of Equation (6.14) reduces to $\ddot{a}_x$, as defined by Equation (6.9).

6-10 Use Equation (6.18) to derive the useful relationship $\bar{A}_x = 1-\delta\cdot\bar{a}_x$.

6-11 Show that $\frac{d}{dx}\bar{a}_x = \bar{a}_x(\mu_x+\delta)-1$.

6-12 Show that if the age-at-failure random variable has a uniform distribution with parameter ω, then the expected value of $\bar{Y}_x$ is given by

$$E[\bar{Y}_x] = \frac{(\omega-x)-\bar{a}_{\overline{\omega-x}|}}{\delta(\omega-x)}.$$

6-13 Find the value of $\bar{A}_x$, given the following values:

$$\bar{a}_x = 10 \qquad {}^2\bar{a}_x = 7.375 \qquad Var(\bar{a}_{\overline{T_x}|}) = 50$$

6-14 Given that $Var(\bar{a}_{\overline{T_x}|}) = \frac{100}{9}$, $\delta = 4k$, and $\mu_{x+t} = k$ for all t, find the value of k.

6-15 If $\delta = .03$ and $\mu_{x+t} = .025$ for all t, calculate the probability that $\bar{Y}_x = \bar{a}_{\overline{T_x}|}$ will exceed 20.

6-16 Show that $Cov(\bar{Y}_x, \bar{Z}_x) = \frac{\bar{A}_x^{\,2} - {}^2\bar{A}_x}{\delta}$.

6-17 $\bar{a}_x$ is calculated using force of failure μ_{x+t} and force of interest δ. $\bar{a}_x^*$ is calculated using force of failure μ_{x+t}^* and force of interest δ^*. Given that $\delta^* = 3\delta$ and $\bar{a}_x^* = \bar{a}_x$ for all x, show that $\mu_{x+t}^* = \mu_{x+t} - 2\delta$.

6-18 A group of persons all age x is made up of 50% females and 50% males. Find the value of $Var(\bar{Y}_x)$ for a person chosen at random from the group, given $\delta = .10$ and the following values:

	Female	Male
$E[\bar{Z}_x]$	0.09	0.15
$Var(\bar{Y}_x)$	4.00	5.00

6.2 Temporary Annuity Models

6-19 Use Equation (6.22b) to show that

$$i \cdot a_{x:\overline{n}|} + i \cdot A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|} = 1.$$

6-20 Repeat the group deterministic demonstration, presented in Section 6.1.1 for the whole life case, to show that the APV and the NSP for the temporary immediate annuity are equal.

6-21 Show that the right side of Equation (6.23) reduces to $a_{x:\overline{n}|}$, as defined by Equation (6.21).

6-22 Repeat Exercise 6-20 to show that the APV and the NSP for the temporary annuity-due are equal.

6-23 Show that the right side of Equation (6.31) reduces to $\ddot{a}_{x:\overline{n}|}$, as defined by Equation (6.27).

6-24 Show that Equation (6.24b) is the same as Equation (6.34) with n replaced by $n+1$.

6-25 Repeat the group deterministic demonstration, presented in Section 6.2.1 for the temporary immediate case, to derive Equation (6.35) for the actuarial accumulated value in the annuity-due case.

6-26 Calculate the value of $\ddot{a}_{x:\overline{4}|}$, given the following values:

k	$\ddot{a}_{\overline{k}|}$	$_{k-1|}q_x$
1	1.00	.33
2	1.93	.24
3	2.80	.16
4	3.62	.11

6-27 Calculate the value of $Var(\ddot{Y}_{x:\overline{3}|})$, given $_tp_x = (.90)^t$ for $t \geq 0$ and the following values of $\ddot{Y}_{x:\overline{3}|}$, for given values of K_x, the random variable for the interval of failure of (x):

K_x	$\ddot{Y}_{x:\overline{3}|}$
1	1.00
2	1.87
3	2.62

6-28 Show that $Var(\bar{Y}_{x:\overline{n}|})$ in terms of annuity functions is given by

$$Var(\bar{Y}_{x:\overline{n}|}) = \frac{2}{\delta}(\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}) - \bar{a}_{x:\overline{n}|}^{\,2}.$$

6-29 Show that

$$\int_0^n \bar{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t}\, dt = \bar{a}_{x:\overline{n}|} - {}_np_x \cdot \bar{a}_{\overline{n}|}.$$

6.3 Deferred Whole Life Annuity Models

6-30 Show that the right side of Equation (6.47) reduces to ${}_{n|}a_x$, as defined by Equation (6.45).

6-31 Show that

$$ {}_{n|}\ddot{Y}_x = \begin{cases} 0 & \text{for } K_x \le n \\ v^n \cdot \ddot{a}_{\overline{K_x - n}|} & \text{for } K_x > n \end{cases}. $$

Then show that

$$ E[{}_{n|}\ddot{Y}_x] = {}_nE_x \cdot \ddot{a}_{x+n} = {}_{n|}\ddot{a}_x. $$

6-32 Let S denote the number of annuity payments actually made under a unit 5-year deferred whole life annuity-due. Find the value of $Pr(S > {}_{5|}\ddot{a}_x)$, given the following values:

$$ \ddot{a}_{x:\overline{5}|} = 4.542 \qquad i = .04 \qquad \mu_{x+t} = .01, \quad \text{for all } t $$

6-33 A 30-year unit deferred whole life annuity-due is issued to (35), with the extra feature that the net single premium is refunded without interest if (35) dies during the deferred period. Calculate the net single premium, given the following values:

$$ \ddot{a}_{65} = 9.90 \qquad A_{35:\overline{30}|} = .21 \qquad A^{1}_{35:\overline{30}|} = .07 $$

6-34 A present value random variable $\bar{Y}$ is defined by

$$ \bar{Y} = \begin{cases} \bar{a}_{\overline{n}|} & \text{for } T_x \le n \\ \bar{a}_{\overline{T_x}|} & \text{for } T_x > n \end{cases}. $$

Show that $E[\bar{Y}] = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$.

6.4 Contingent Annuities Payable m^{thly}

6-35 Derive each of Equations (6.66) through (6.70).

6-36 (a) Show that $\ddot{a}^{(m)}_{x:\overline{n}|} = a^{(m)}_{x:\overline{n}|} + \frac{1}{m}(1 - {}_nE_x)$.

(b) Show that ${}_{n|}\ddot{a}^{(m)}_x = {}_{n|}a^{(m)}_x + \frac{1}{m} \cdot {}_nE_x$.

6-37 Derive Equation (6.77).

6-38 Derive the following UDD-based approximations.

(a) ${}_{n|}\ddot{a}^{(m)}_x \approx \frac{id}{i^{(m)}d^{(m)}} \cdot {}_{n|}\ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} \cdot {}_nE_x$

(b) $\ddot{a}^{(m)}_{x:\overline{n}|} \approx \frac{id}{i^{(m)}d^{(m)}} \cdot \ddot{a}_{x:\overline{n}|} - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}(1 - {}_nE_x)$

(c) $\ddot{s}^{(m)}_{x:\overline{n}|} \approx \frac{id}{i^{(m)}d^{(m)}} \cdot \ddot{s}_{x:\overline{n}|} - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}\left(\frac{1}{{}_nE_x} - 1\right)$

6-39 Derive the following UDD-based approximations.

(a) $a^{(m)}_x \approx \frac{id}{i^{(m)}d^{(m)}} \cdot a_x + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}}$

(b) ${}_{n|}a^{(m)}_x \approx \frac{id}{i^{(m)}d^{(m)}} \cdot {}_{n|}a_x + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}} \cdot {}_nE_x$

(c) $a^{(m)}_{x:\overline{n}|} \approx \frac{id}{i^{(m)}d^{(m)}} \cdot a_{x:\overline{n}|} + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}}(1 - {}_nE_x)$

(d) $s^{(m)}_{x:\overline{n}|} \approx \frac{id}{i^{(m)}d^{(m)}} \cdot s_{x:\overline{n}|} + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}}\left(\frac{1}{{}_nE_x} - 1\right)$

6-40 (a) By taking the limit as $m \to \infty$ in Equation (6.78), derive the UDD-based approximation

$$\bar{a}_x \approx \frac{id}{\delta^2} \cdot \ddot{a}_x - \frac{i-\delta}{\delta^2}.$$

(b) By taking the limit as $m \to \infty$ in Exercise 6-39(a), derive the alternative UDD-based approximation

$$\bar{a}_x \approx \frac{id}{\delta^2} \cdot a_x + \frac{\delta - d}{\delta^2}.$$

(c) Show that the right sides in parts (a) and (b) are equal.

6-41 An alternative (and simpler) approximation for $\ddot{a}_x^{(m)}$ is

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}.$$

From this approximation, derive each of the following:

(a) $${}_{n|}\ddot{a}_x^{(m)} \approx {}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x$$

(b) $$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

(c) $$\ddot{s}_{x:\overline{n}|}^{(m)} \approx \ddot{s}_{x:\overline{n}|} - \frac{m-1}{2m}\left(\frac{1}{{}_nE_x} - 1\right)$$

(d) $$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

(e) $${}_{n|}a_x^{(m)} \approx {}_{n|}a_x + \frac{m-1}{2m} \cdot {}_nE_x$$

(f) $$a_{x:\overline{n}|}^{(m)} \approx a_{x:\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x)$$

(g) $$s_{x:\overline{n}|}^{(m)} \approx s_{x:\overline{n}|} + \frac{m-1}{2m}\left(\frac{1}{{}_nE_x} - 1\right)$$

6-42 Show that the alternative approximation of Exercise 6-41, applied to continuous annuity models, leads to

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} = a_x + \frac{1}{2}.$$

6-43 Show that, under the UDD assumption,

$$\bar{A}_x = \frac{i}{\delta} - \frac{(i-d)\ddot{a}_x}{\delta}.$$

6.5 Non-Level Payment Annuity Functions

6-44 Calculate the probability that the present value of payments actually made under a unit 3-year temporary increasing annuity-due will exceed the APV of the annuity contract, given the following values:

$$p_x = .80 \qquad p_{x+1} = .75 \qquad p_{x+2} = .50 \qquad v = .90$$

6-45 An increasing temporary annuity-due pays 2 in the first year, 3 in the second year, and 4 in the third year. Using the values given in Exercise 6-44, calculate the variance of the present value random variable for this annuity.

6-46 (a) Show that $(IA)_x = \ddot{a}_x - d \cdot (I\ddot{a})_x$.

(b) Show that $(\bar{I}\bar{A})_x = \bar{a}_x - \delta \cdot (\bar{I}\bar{a})_x$.