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CHAPTER FIVE

CONTINGENT PAYMENT MODELS (INSURANCE MODELS)

5-1 From Equation (5.18) we have

$$Cov(Z^1_{x:\overline{n|}},\ {}_{n|}Z_x) = \frac{1}{2}\Big[Var(Z_x) - Var(Z^1_{x:\overline{n|}}) - Var({}_{n|}Z_x)\Big]$$
$$= \frac{1}{2}\Big[{}^2A_x - A_x^{\,2} - {}^2A^1_{x:\overline{n|}} + A^{1\ 2}_{x:\overline{n|}} - {}^2{}_{n|}A_x + {}_{n|}A_x^{\,2}\Big]$$
$$= \frac{1}{2}\Big[A^{1\ 2}_{x:\overline{n|}} + {}_{n|}A_x^{\,2} - A_x^{\,2}\Big],$$

since ${}^2A_x - {}^2A^1_{x:\overline{n|}} - {}^2{}_{n|}A_x = 0$ by Equation (5.17).

5-2 We start with the right-hand side of Equation (5.19a) and substitute for A_x from Equation (5.16), obtaining

$$\frac{1}{2}\Big[A^{1\ 2}_{x:\overline{n|}} + {}_{n|}A_x^{\,2} - (A^1_{x:\overline{n|}} + {}_{n|}A_x)^2\Big]$$
$$= \frac{1}{2}\Big[A^{1\ 2}_{x:\overline{n|}} + {}_{n|}A_x^{\,2} - (A^{1\ 2}_{x:\overline{n|}} + 2\cdot A^1_{x:\overline{n|}}\cdot {}_{n|}A_x + {}_{n|}A_x^{\,2})\Big]$$
$$= -A^1_{x:\overline{n|}}\cdot {}_{n|}A_x, \text{ as required.}$$

5-3 From Equation (5.19b) we directly have

$$Cov(Z^1_{x:\overline{3|}},\ {}_{3|}Z_x) = -A^1_{x:\overline{3|}}\cdot {}_{3|}A_x$$
$$= -(.686227)(.286867)$$
$$= -.19686,$$

where the values of $A^1_{x:\overline{3|}}$ and ${}_{3|}A_x$ are found in Examples 5.2 and 5.3, respectively.

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5-4 When $i=0$ the present value factor is $v=1$, so the present value random variable $Z=1$ as well. Then

$$E[Z] = 1\cdot q_x + 1\cdot p_x \cdot q_{x+1} = .50 + .50q_{x+1}.$$

The second moment is the same (since $Z=1$), so the variance is

$$\begin{aligned} Var(Z) &= (.50+.50q_{x+1}) - (.50+.50q_{x+1})^2 \\ &= (.50+.50q) - (.25+.50q+.25q^2) \\ &= .25 - .25q^2 = .1771, \end{aligned}$$

which solves for $q = q_{x+1} = .54$.

5-5 The present value of payment is $b\cdot v$ if (x) fails, which happens with probability q_x, or $e\cdot v$ if (x) survives, which happens with probability p_x. Then we have

$$E[Z] = bv\cdot q_x + ev\cdot p_x$$

and

$$E[Z^2] = b^2v^2\cdot q_x + e^2v^2\cdot p_x,$$

so

$$\begin{aligned} Var(Z) &= b^2v^2\cdot q_x + e^2v^2\cdot p_x - (bv\cdot q_x + ev\cdot p_x)^2 \\ &= b^2v^2\cdot q_x + e^2v^2\cdot p_x - b^2v^2\cdot q_x{}^2 - 2be\cdot v^2\cdot q_x\cdot p_x - e^2v^2\cdot p_x{}^2 \\ &= b^2v^2\cdot q_x(1-q_x) + e^2v^2\cdot p_x(1-p_x) - 2be\cdot v^2\cdot q_x\cdot p_x \\ &= b^2v^2\cdot q_x\cdot p_x - 2be\cdot v^2\cdot q_x\cdot p_x + e^2v^2\cdot q_x\cdot p_x \\ &= v^2(b-e)^2\cdot q_x\cdot p_x, \text{ as required.} \end{aligned}$$

5-6 From Equation (5.2),

$$A_x = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_x = v \cdot q_x + \sum_{k=2}^{\infty} v^k \cdot {}_{k-1|}q_x.$$

Letting $j = k-1$ we then have

$$\begin{aligned} A_x &= v \cdot q_x + \sum_{j=1}^{\infty} v^{j+1} \cdot {}_{j|}q_x = v \cdot q_x + v \cdot p_x \sum_{j=1}^{\infty} v^j \cdot {}_{j-1|}q_{x+1} \\ &= v \cdot q_x + v \cdot p_x \cdot A_{x+1}, \text{ as required.} \end{aligned}$$

5-7 Recall from Exercise 5-6 that $A_{50} = v \cdot q_{50} + v \cdot p_{50} \cdot A_{51}$. Then

$$\begin{aligned} A_{51} - A_{50} &= A_{51} - v \cdot q_{50} - v \cdot p_{50} \cdot A_{51} \\ &= A_{51}(1 - v \cdot p_{50}) - v \cdot q_{50} \\ &= A_{51}\left(1 - \frac{.98}{1.02}\right) - \frac{.02}{1.02} = .004, \end{aligned}$$

which implies $A_{51} = .60199$.

Similarly,

$$\begin{aligned} {}^2A_{51} - {}^2A_{50} &= {}^2A_{51} - v' \cdot q_{50} - v' \cdot p_{50} \cdot {}^2A_{51} \\ &= {}^2A_{51}(1 - v' \cdot p_{50}) - v' \cdot q_{50} \\ &= {}^2A_{51}\left(1 - \frac{.98}{(1.02)^2}\right) - \frac{.02}{(1.02)^2} = .005, \end{aligned}$$

which implies ${}^2A_{51} = .41725$.

Then

$$Var(Z_{51}) = {}^2A_{51} - A_{51}^2 = .41725 - (.60199)^2 = .05486.$$

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5-8 Recall that

$$Var(Z_1 + Z_2) = Var(Z_1) + Var(Z_2) + 2 \cdot Cov(Z_1, Z_2),$$

and, in turn,

$$Cov(Z_1, Z_2) = E[Z_1 \cdot Z_2] - E[Z_1] \cdot E[Z_2].$$

The insurance represented by Z_1 pays only for failure in $(0, 25]$, and the insurance represented by Z_2 pays only for failure in $(25, 35]$. Therefore for all K_x one or the other of Z_1 and Z_2 must be zero, so $E[Z_1 \cdot Z_2] = 0$.

Therefore

$$\begin{aligned} Var(Z_1 + Z_2) &= Var(Z_1) + Var(Z_2) - 2 \cdot E[Z_1] \cdot E[Z_2] \\ &= 5.76 + .10 - 2(2.80)(.12) = 5.188. \end{aligned}$$

5-9 Recall that $Var(Z) = 1000^2 ({}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}{}^2)$.

Next recall that

$$\begin{aligned} {}^2A_{x:\overline{n}|} &= {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1} \\ &= {}^2A_x - {}^2{}_{n|}A_x + {}^2A_{x:\overline{n}|}^{\ \ 1} \\ &= {}^2A_x - {}^2A_{x:\overline{n}|}^{\ \ 1} \cdot {}^2A_{x+n} + {}^2A_{x:\overline{n}|}^{\ \ 1} \\ &= .2196 - (.5649)(.2836) + .5649 \\ &= .6243. \end{aligned}$$

Then

$$Var(Z) = 1000^2 \left(.6243 - (.7896)^2\right) = 831.84.$$

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5-10 We are given

$$Z_1 = 1000 \cdot {}_{10}E_x \cdot Z^{1}_{x:\overline{10}|} + 1000 \cdot Z_{x:\overline{10}|}^{\ \ \ 1},$$

so

$$E[Z_1] = 1000 \cdot {}_{10}E_x \cdot A^{1}_{x:\overline{10}|} + 1000 \cdot {}_{10}E_x.$$

Similarly,

$$E[Z_2] = 750 \cdot {}_{10}E_x \cdot A^{1}_{x:\overline{10}|} + 1000 \cdot {}_{10}E_x$$

and

$$E[Z_3] = 500 \cdot {}_{10}E_x \cdot A^{1}_{x:\overline{10}|} + 1000 \cdot {}_{10}E_x.$$

Then

$$\frac{E[Z_1]}{E[Z_2]} = \frac{1000A^{1}_{x:\overline{10}|} + 1000}{750A^{1}_{x:\overline{10}|} + 1000} = 1.005,$$

which solves for $A^{1}_{x:\overline{10}|} = \frac{5}{246.25} = .02030$. Next we find

$${}_{10}E_x = A_{x:\overline{10}|}^{\ \ \ 1} = A_{x:\overline{10}|} - A^{1}_{x:\overline{10}|} = .5700 - .0203 = .5497.$$

Finally we have

$$E[Z_3] = (500)(.5497)(.0203) + (1000)(.5497) = 555.27.$$

5-11 From Equation (5.12) we have

$${}_{n|}A_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1|}q_x = \sum_{k=n+1}^{\infty} v^k \cdot \frac{d_{x+k-1}}{\ell_x}.$$

Let $j = k-n$ so $k = j+n$. Then

$${}_{n|}A_x = \sum_{j=1}^{\infty} v^{j+n} \cdot \frac{d_{x+j+n-1}}{\ell_x} = v^n \cdot \frac{\ell_{x+n}}{\ell_x} \sum_{j=1}^{\infty} v^j \cdot \frac{d_{x+n+j-1}}{\ell_{x+n}}$$

$$= {}_nE_x \cdot A_{x+n}, \text{ as required.}$$

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5-12 (a) From Equation (5.8), we have

$$A^{1}_{x:\overline{n}|} = \sum_{t=1}^{n} v^t \cdot {}_{t-1|}q_x$$

$$= \sum_{t=1}^{n} v^t \cdot \frac{d_{x+t-1}}{\ell_x}$$

$$= v \cdot \frac{d_x}{\ell_x} + v^2 \cdot \frac{d_{x+1}}{\ell_x} + \cdots + v^n \cdot \frac{d_{x+n-1}}{\ell_x},$$

which gives the NSP for a unit insurance with benefit paid at the end of the year of failure provided failure occurs *within* the first n years.

(b) From Equation (5.12),

$$_{n|}A_x = \sum_{t=n+1}^{\infty} v^t \cdot {}_{t-1|}q_x = v^{n+1} \cdot \frac{d_{x+n}}{\ell_x} + v^{n+2} \cdot \frac{d_{x+n+1}}{\ell_x} + \cdots,$$

the NSP for a unit insurance for failure occurring *after* the first n years.

(c) Here we have

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$$

$$= v \cdot \frac{d_x}{\ell_x} + v^2 \cdot \frac{d_{x+1}}{\ell_x} + \cdots + v^n \cdot \frac{d_{x+n-1}}{\ell_x} + v^n \cdot \frac{\ell_{x+n}}{\ell_x},$$

where the last term represents the present value of benefits paid to those who survive to age $x+n$.

5-13 Using the identity from Exercise 5-6 we have

$$A_{76} = v \cdot q_{76} + v \cdot p_{76} \cdot A_{77}$$

$$= v - v \cdot p_{76} + v \cdot p_{76} \cdot A_{77} = v - v \cdot p_{76}(1 - A_{77}).$$

Substituting the given values we have

$$.800 = \frac{1}{1.03} - (.90)(1 - A_{77}),$$

which solves for $A_{77} = .81014$.

5-14 We are given that

$$\begin{aligned} 600 &= 600A^{1}_{x:\overline{n}|} + 1000A_{x:\overline{n}|}^{\;\;1} \\ &= 600\left(A_{x:\overline{n}|} - A_{x:\overline{n}|}^{\;\;1}\right) + 1000A_{x:\overline{n}|}^{\;\;1} \\ &= 600A_{x:\overline{n}|} + 400A_{x:\overline{n}|}^{\;\;1}, \end{aligned}$$

so

$$A_{x:\overline{n}|}^{\;\;1} = {}_nE_x = \frac{600(1-A_{x:\overline{n}|})}{400} = \frac{(600)(.20)}{400} = .30.$$

5-15 Using Equation (5.35) we have

$$\begin{aligned} APV &= 50\int_0^{100} v^t \cdot f(t)\ dt \\ &= \frac{50}{5000}\int_0^{100} \begin{array}{c|c} t\cdot & e^{-.10t}\ dt \\ dt & \frac{e-.10t}{-.10} \end{array} \\ &= \frac{1}{100}\left[\frac{t}{-.10}\cdot e^{-.10t}\Big|_0^{100} + 10\int_0^{100} e^{-.10t}\ dt\right] \\ &= \frac{1}{100}\left[-1000e^{-10} + \frac{10}{-.10}\cdot e^{-.10t}\Big|_0^{100}\right] \\ &= \frac{1}{100}\left[-1000e^{-10} + 100(1-e^{-10})\right] \\ &= 1-11e^{-10} = .99950. \end{aligned}$$

5-16 (a) The random variable $\bar{Z}^{1}_{x:\overline{n}|}$ denotes the present value of a unit paid at the instant of failure, provided failure occurs in $(0,n]$. If failure occurs at time T_x, the present value is v^{T_x}, but only for $T_x \leq n$. If $T_x > n$, then the present value is zero.

(b) Since $\bar{Z}^{1}_{x:\overline{n}|}$ is a function of the continuous random variable T_x, its moments are found by integrating powers of $\bar{Z}^{1}_{x:\overline{n}|}$ times the PDF of T_x, which is given by ${}_tp_x\mu_{x+t}$.

5-17 (a) Payment is made at the moment of failure provided failure occurs after time n. Thus the present value is v^{T_x} for $T_x > n$. If $T_x \leq n$, then the present value is zero.

(b) Again the moments are found by integrating powers of ${}_{n|}\overline{Z}_x$ times the PDF of T_x, which is ${}_t p_x \mu_{x+t}$.

(c) From the definitions of $\overline{Z}^{1}_{x:\overline{n}|}$ and ${}_{n|}\overline{Z}_x$ given in Exercises 5-16(a) and 5-17(a), respectively, it is easy to see that

$$\overline{Z}_x = \overline{Z}^{1}_{x:\overline{n}|} + {}_{n|}\overline{Z}_x.$$

Taking the expectation on both sides produces the given relationship.

5-18 The random variable ${}_{20|}\overline{Z}_{40}$ takes on the discrete value zero whenever $T_{40} \leq 20$. Thus the discrete part of the distribution is ${}_{20|}\overline{Z}_{40} = 0$ with probability mass given by

$$Pr(T_{40} \leq 20) = \int_0^{20} (110-40)^{-1}\, dt = \frac{1}{70} t \Big|_0^{20} = \frac{2}{7}.$$

5-19 Recall that $\overline{Z}_{x:\overline{n}|} = \overline{Z}^{1}_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$, so

$$Var(\overline{Z}_{x:\overline{n}|}) = Var(\overline{Z}^{1}_{x:\overline{n}|}) + Var(Z_{x:\overline{n}|}^{\ \ 1}) + 2 \cdot Cov(\overline{Z}^{1}_{x:\overline{n}|}, Z_{x:\overline{n}|}^{\ \ 1}).$$

First we find

$$\begin{aligned} Var(Z_{x:\overline{n}|}^{\ \ 1}) &= {}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2 \\ &= (v^n)^2 \cdot {}_n p_x - (v^n \cdot {}_n p_x)^2 \\ &= (.20)^2(.50) - \left[(.20)(.50)\right]^2 = .01. \end{aligned}$$

From Equation (5.29) we have

$$\begin{aligned} Cov(\overline{Z}^{1}_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1}) &= -\overline{A}^{1}_{x:\overline{n}|} \cdot A_{x:\overline{n}|}^{\ \ 1} \\ &= -(.23)(.20)(.50) = -.023. \end{aligned}$$

Then

$$Var(\overline{Z}_{x:\overline{n}|}) = .08 + .01 + 2(-.023) = .044.$$

5-20 It follows from Equation (5.44) that

$$ {}^{2}\bar{A}_x = E[\bar{Z}_x{}^2] = \frac{\lambda}{2\delta+\lambda}. $$

Then we have

$$ Var(\bar{Z}_x) = \frac{\lambda}{2\delta+\lambda} - \left(\frac{\lambda}{\delta+\lambda}\right)^2. $$

5-21 For the deferred insurance we have

$$ {}_{n|}\bar{A}_x = E\left[{}_{n|}\bar{Z}_x\right] = \int_n^\infty e^{-\delta t} \cdot e^{-\lambda t} \cdot \lambda \, dt $$

$$ = \lambda\left(\frac{e^{-(\delta+\lambda)t}}{-(\delta+\lambda)}\right)\Bigg|_n^\infty = \frac{\lambda}{\delta+\lambda} \cdot e^{-(\delta+\lambda)n}. $$

5-22 Since $\bar{Z}_{40} = v^{T_{40}}$, we have the transformation $z = v^t$ so $t = \frac{-\ln z}{\delta}$. The transformation is decreasing so we have

$$ F_{Z_{40}}(z) = S_{T_{40}}\left[-\frac{\ln z}{\delta}\right]. $$

But $S_{T_{40}}(t) = {}_tp_{40} = 1 - \frac{t}{\omega - 40} = 1 - \frac{t}{70}$ under a uniform distribution with $\omega = 110$. Therefore,

$$ F_{Z_{40}}(z) = 1 - \frac{-\frac{\ln z}{\delta}}{70} = 1 + \frac{\ln z}{3.50}, $$

since $\delta = .05$. Then

$$ f_{Z_{40}}(z) = \frac{d}{dz} F_{Z_{40}}(z) = \frac{1}{3.50z}, $$

and finally

$$ f_{Z_{40}}(.80) = \frac{1}{(3.50)(.80)} = .35714. $$

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5-23 Recall (see Equation (2.13)) that the moment generating function is defined as $M_{T_x}(r) = E[e^{r \cdot T_x}]$. Then we have

$$\begin{aligned} M_{T_x}(-\delta) &= E[e^{-\delta \cdot T_x}] \\ &= \int_0^\infty e^{-\delta t} \cdot f_{T_x}(t)\, dt \\ &= \int_0^\infty v^t \cdot {}_tp_x \mu_{x+t}\, dt, \end{aligned}$$

which is $\bar{A}_x$ by Equation (5.35).

5-24 The expected value of $\bar{B}_x$ is

$$\begin{aligned} E[\bar{B}_x] &= \int_0^{50} b_t \cdot v^t \cdot f(t)\, dt \\ &= .02 \int_0^{50} (1+.10t) \cdot (1+.10t)^{-2}\, dt \\ &= .02 \int_0^{50} (1+.10t)^{-1}\, dt = .02 \left[\frac{\ln(1+.10t)}{.10} \Big|_0^{50} \right] \\ &= .02 \left[\frac{\ln 6}{.10} \right] = .358352. \end{aligned}$$

The second moment is

$$\begin{aligned} E[\bar{B}_x{}^2] &= \int_0^{50} (b_t \cdot v^t)^2 \cdot f(t)\, dt \\ &= .02 \int_0^{50} (1+.10t)^{-2}\, dt \\ &= .02 \left[\frac{(1+.10t)^{-1}}{-.10} \Big|_0^{50} \right] \\ &= .02 \left[\frac{1-6^{-1}}{.10} \right] = .166667. \end{aligned}$$

Then the variance is

$$Var(\bar{B}_x) = .166667 - (.358352)^2 = .03825.$$

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5-25 (a) From Equation (5.51) we have

$$(IA)_x = \sum_{k=1}^{\infty} k \cdot v^k \cdot {}_{k-1|}q_x = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_x + \sum_{k=2}^{\infty} (k-1) \cdot v^k \cdot {}_{k-1|}q_x.$$

Let $r = k-1$ so $k = r+1$. Then

$$(IA)_x = A_x + \sum_{r=1}^{\infty} r \cdot v^{r+1} \cdot {}_{r|}q_x$$

$$= A_x + v \cdot p_x \sum_{r=1}^{\infty} r \cdot v^r \cdot {}_{r-1|}q_{x+1} = A_x + {}_1E_x \cdot (IA)_{x+1}.$$

(b) Note that $A_{35:\overline{1}|} = v = .9434$. Then

$${}_1E_{35} = v \cdot p_{35} = (.9434)(.9964) = .94000.$$

Using the result from part (a), we have

$$(IA)_{35} = A_{35} + {}_1E_{35} \cdot (IA)_{36}.$$

Then

$$(IA)_{36} = \frac{(IA)_{35} - A_{35}}{{}_1E_{35}} = \frac{3.711 - .130}{.940} = 3.80957.$$

5-26 First we need to find the parameters of the Pareto survival model. If $E[X] = 4$ and $Var(X) = 48$, then $E[X^2] = 48+16 = 64$. From Equation (12.3) we have $\frac{\theta}{\alpha - 1} = 4$, so $\theta = 4(\alpha-1)$. From Equation (12.4) we have

$$\frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{2(16)(\alpha-1)^2}{(\alpha-1)(\alpha-2)} = \frac{32(\alpha-1)}{\alpha - 2} = 64,$$

which solves for $\alpha = 3$ and therefore $\theta = 8$. Then we have $S(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha} = \left(\frac{8}{8+x}\right)^3$, so $S(1) = \left(\frac{8}{9}\right)^3$, $S(2) = \left(\frac{8}{10}\right)^3$, and

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$S(3) = \left(\frac{8}{11}\right)^3$. The warranty benefits are 1500, 1000, and 500, for failure in years 1, 2, 3, respectively. Then the APV is

$$APV = \frac{1500}{1.05}\left[1-\left(\frac{8}{9}\right)^3\right] + \frac{1000}{(1.05)^2}\left[\left(\frac{8}{9}\right)^3 - \left(\frac{8}{10}\right)^3\right] + \frac{500}{(1.05)^3}\left[\left(\frac{8}{10}\right)^3 - \left(\frac{8}{11}\right)^3\right]$$

$$= .425.24 + 172.64 + 54.99 = 652.87.$$

5-27 The rate of loan payment is

$$P = \frac{10,000}{\bar{a}_{\overline{20}|\delta=.08}} = \frac{10,000}{\frac{1-e^{-(.08)(20)}}{.08}} = 1002.38.$$

The outstanding balance at time t, which is the benefit under the contingent payment contract is

$$b_t = P \cdot \bar{a}_{\overline{20-t}|\delta=.08} = \frac{1002.38}{.08}(1-e^{-.08(20-t)}) = 12,529.70(1-e^{-1.6} \cdot e^{.08t}).$$

The APV is given by

$$APV = \int_0^{20} b_t \cdot v^t \cdot {}_tp_0 \cdot \mu_t \, dt$$

$$= (12,529.70)(.01)\int_0^{20}(1-e^{-1.6} \cdot e^{.08t}) \cdot e^{-.05t} \cdot e^{-.01t}\, dt$$

$$= 125.30\left[\int_0^{20} e^{-.06t}\, dt - e^{-1.6}\int_0^{20} e^{.02t}\, dt\right]$$

$$= 125.30\left[\frac{1-e^{-1.2}}{.06} - e^{-1.6}\left(\frac{e^{.40}-1}{.02}\right)\right] = 837.24.$$