SOLUTIONS TO UNIT REVIEW QUESTIONS

S-1. $F_X(75) = 1 - s(75) = 1 - \frac{\sqrt{100-75}}{10} = 1 - \frac{1}{2} = .5.$

Now $f_X(x) = -s'_X(x) = -\frac{1}{10} \cdot \frac{1}{2} \frac{-1}{\sqrt{100-x}}$; $f_X(75) = \frac{1}{20\sqrt{100-75}} = \frac{1}{100} = .01.$

Also $f_X(75) = s_X(75)\mu(75)$ so $\mu(75) = \frac{f_X(75)}{s_X(75)} = \frac{1/100}{1/2} = \frac{1}{50} = .02.$ ANSWER B

S-2. Consider the table below based on ${}_{1|}q_{x+1} = .095$ and $.171 = {}_{2|}q_{x+1}$. Then from $.200 = q_{x+3} = \frac{171}{\ell_{x+3}}$ we obtain $\ell_{x+3} = 855$. Thus $\ell_{x+2} = 855 + 95 = 950$, and $d_{x+1} = 1000 - 950 = 50$. Hence $q_{x+1} + q_{x+2} = \frac{50}{1000} + \frac{95}{950} = .05 + .10 = .15$

age	living	deaths
$x+1$	1000	
$x+2$	↑	95
$x+3$	ℓ_{x+3}	171

ANSWER A

S-3.

$$K = {}_{1/3}q_x \text{ (Balducci)} = \frac{\ell_x - \ell^{Bal}_{x+1/3}}{\ell_x} = \frac{9 - \frac{54}{7}}{9} = \frac{9}{63}$$

$$L = {}_{1/3|2/3}q_x \text{ (UDD)} = \frac{\ell^{UDD}_{x+1/3} - \ell_{x+1}}{\ell_x} = \frac{8-6}{9} = \frac{2}{9} = \frac{14}{63}$$

Sum $K + L = \frac{9+14}{63} = \frac{23}{63}$ ANSWER C

Balducci

$$\frac{1}{\ell_{x+1/3}} = \frac{2}{3} \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{6} = \frac{2}{27} + \frac{1}{18} = \frac{4+3}{54}, \quad \ell_{x+1/3} = \frac{54}{7}$$

UDD

$$\ell_{x+1/3} = \frac{2}{3} \cdot 9 + \frac{1}{3} \cdot 6 = 8$$

S-4. $E[K] = \frac{\ell_{97} + \ell_{98} + \ell_{99}}{\ell_{96}} = \frac{130 + 73 + 31}{180} = \frac{234}{180} = 1.3$

$$E[K^2] = \frac{1 \cdot \ell_{97} + 3 \cdot \ell_{98} + 5 \cdot \ell_{99}}{\ell_{96}} = \frac{130 + 219 + 155}{180} = 2.8$$

$Var(K) = 2.8 - 1.3^2 = 1.11$ ANSWER D

S-5. With de Moivre's Law $T \sim U(0, \omega - x) = U(0, \omega - 16)$.

In this case $42 = \overset{\circ}{e}_x = E[T] = \frac{\omega - 16}{2}$ implies $\omega = 100$.
Also $Var(T) = \frac{(\omega - 16)^2}{12} = \frac{84^2}{12} = 588$ for a uniform distribution. ANSWER B

S-6. $q_{50} = \frac{\ell_{50} - \ell_{51}}{\ell_{50}} = \frac{s_X(50) - s_X(51)}{s_X(50)} = \frac{.66\bar{6} - .654\bar{3}}{.66\bar{6}} = .0185$

$\mu_{50} = \frac{-s'_X(50)}{s_X(50)} = \frac{(10 + 2 \cdot 50)/9000}{.66\bar{6}} = .018\bar{3}$

$q_{50} - \mu_{50} = .000167 = \frac{1}{6000}$ ANSWER D

S-7. $E[T] = \int_0^{100-x} {}_tp_x \, dt = \int_0^{100-x} \left(\frac{100 - x - t}{100 - x}\right)^2 dt$

$$= \frac{-\frac{1}{3}(100 - x - t)^3 \Big|_{t=0}^{100-x}}{(100 - x)^2} = \frac{1}{3}(100 - x)$$

$$E[T^2] = \int_0^{100-x} 2t \, {}_tp_x \, dt = \int_0^{100-x} \frac{2t(100 - x - t)^2}{(100 - x)^2} dt$$

$$= -2\int_0^{100-x} \frac{-t(100 - x - t)^2}{(100 - x)^2} dt = -2\int_0^{100-x} \frac{[(100 - x - t) - (100 - x)](100 - x - t)^2}{(100 - x)^2} dt$$

$$= \frac{-2}{(100 - x)^2}\left[\int_0^{100-x} (100 - x - t)^3 \, dt - (100 - x)\int_0^{100-x} (100 - x - t)^2 \, dt\right]$$

$$= \frac{-2}{(100 - x)^2}\left[\frac{(100 - x)^4}{4} - \frac{(100 - x)^4}{3}\right] = \frac{(100 - x)^2}{6}.$$

$$Var(T) = E[T^2] - \left(E[T]\right)^2 = (100 - x)^2\left(\frac{1}{6} - \left(\frac{1}{3}\right)^2\right) = \frac{(100 - x)^2}{18}$$ ANSWER A

This problem is a special case of the more general situation where $\mu(x) = \frac{n}{\omega - x}$, $E[T] = \frac{\omega - x}{n + 1}$, and $Var(T) = \frac{(\omega - x)^2 \cdot n}{(n + 1)^2 \cdot (n + 2)}$

S-8. ${}_{20|10}q_5 = Pr(5 \text{ dies between ages 25 and 35}) = \frac{s_X(25) - s_X(35)}{s_X(5)}$.

$s_X(x) = e^{-\int_0^x \mu(s) ds} = e^{-\int_0^x s/100 \, ds} = e^{-x^2/200}$; $s_X(25) = e^{-625/200}$,
$s_X(35) = e^{-1225/200}$, $s_X(5) = e^{-25/200}$. Thus

$${}_{20|10}q_5 = \frac{e^{-625/200} - e^{-1225/200}}{e^{-25/200}} = e^{-600/200} - e^{-1200/200} = e^{-3} - e^{-6} = \frac{e^3 - 1}{e^6}$$

ANSWER C

S-9. We are given $q_{[x+1]} = \frac{2}{3} q_{[x]+1}$, $q_{[x+1]+1} = \frac{3}{4} q_{x+2}$. To find $\ell_{[94]}$ we must find ${}_2p_{[94]} = p_{[94]} \cdot p_{[94]+1} = (1 - q_{[94]})(1 - q_{[94]+1})$.

Now $q_{[94]+1} = \frac{3}{4} \cdot q_{95} = \frac{3}{4} \cdot \frac{5040 - 3024}{5040} = \frac{3}{4} \cdot \frac{2}{5} = \frac{3}{10}$ and

$q_{[94]} = \frac{2}{3} \cdot q_{[93]+1} = \frac{2}{3} \cdot \frac{3}{4} \cdot q_{94} = \frac{1}{2} \cdot \frac{6300 - 5040}{6300} = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$.

Thus $\left(1 - \frac{3}{10}\right)\left(1 - \frac{1}{10}\right) = \frac{7}{10} \cdot \frac{9}{10} = {}_2p_{[94]} = \frac{\ell_{96}}{\ell_{[94]}} = \frac{3024}{\ell_{[94]}}$.

Hence $\ell_{[94]} = \frac{100}{63} \cdot 3024 = 4800$. ANSWER A

S-10. ${}_tp_x = e^{-\int_x^{x+t} \mu(s)\,ds}$. Also $\int_x^{x+t} \mu(s)\,ds = \int_x^{x+t} ks\,ds = \frac{k}{2} s^2 \Big|_x^{x+t} = \frac{k}{2}(2tx + t^2)$.

Thus $.81 = {}_{10}p_{35} = e^{-k/2\,(2 \cdot 10 \cdot 35 + 10^2)} = e^{-400k}$. Also

$${}_{20}p_{40} = e^{-(k/2)(2 \cdot 20 \cdot 40 + 20^2)} = e^{-1000k} = \left(e^{-400k}\right)^{5/2} = .81^{5/2}$$

$$= (.81)^2(.9) = \left(\frac{9}{10}\right)^5 = .59049.$$

ANSWER D

S-11. If ${}_{t|}q_x = .10$ for $t = 0, 1, \ldots, 9$, an obvious choice for ℓ_{x+t} is $10 - t$. Thus

$${}_2p_{x+5} = \frac{\ell_{x+7}}{\ell_{x+5}} = \frac{10 - 7}{10 - 5} = \frac{3}{5}$$

ANSWER B

S-12. I. ${}_{1|2}q_{36} = Pr[(36)$ dies between ages 37 and 39$] = \frac{\ell_{37} - \ell_{39}}{\ell_{36}} = \frac{96 - 87}{99} = \frac{1}{11} = .091$, *true*.

II. $m_{37} = \frac{d_{37}}{L_{37}} \underset{\text{UDD}}{=} \frac{d_{37}}{\ell_{37} - \frac{1}{2} d_{37}} = \frac{4}{96 - \frac{1}{2} \cdot 4} = \frac{4}{94} = .043$, *true*.

III. ${}_{.33}q_{38.5} = \frac{\ell_{38.5} - \ell_{38.83}}{\ell_{38.5}} = \frac{(92 - \frac{1}{2} \cdot 5) - (92 - \frac{5}{6} \cdot 5)}{92 - \frac{1}{2} \cdot 5} = \frac{5/3}{89.5} = .0186$, *false*.

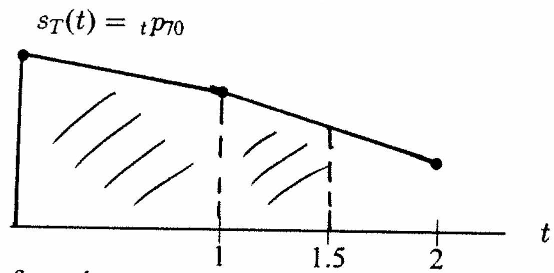
ANSWER A

S-13. Under the UDD assumption $\overset{\circ}{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + \frac{1}{2} \cdot {}_nq_x$. So $\overset{\circ}{e}_{45:\overline{1}|} = e_{45:\overline{1}|} + \frac{1}{2} \cdot q_{45} = p_{45} + \frac{1}{2} \cdot q_{45}$ since $e_{x:\overline{n}|} = {}_1p_x + {}_2p_x + \cdots + {}_np_x$. Also, from the UDD assumption, $\mu(45.5) = \frac{q_{45}}{1 - \frac{1}{2} \cdot q_{45}}$, so $\frac{1}{2} = \frac{q_{45}}{1 - \frac{1}{2} \cdot q_{45}}$ implies $q_{45} = \frac{1}{2.5}$. Finally $\overset{\circ}{e}_{45:\overline{1}|} = p_{45} + \frac{1}{2} \cdot q_{45} = \frac{1.5}{2.5} + \frac{.5}{2.5} = .8$, ANSWER E.

ALTERNATE METHOD:

Use $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt$ and the facts that $n = 1$ and ${}_tp_x = 1 - {}_tq_x = 1 - t \cdot q_x$ under UDD.

S-14. The starting point is to note that $\overset{\circ}{e}_{70:\overline{1.5}|} = \int_0^{1.5} {}_tp_{70} \cdot dt$. From the given $q_{70} = .040$ and $q_{71} = .044$, and the UDD assumption, we obtain $p_{70} = .960$, ${}_2p_{70} = (.960)(.956) = .91584$ and the piecewise linear picture below:



The corresponding formula,

$${}_tp_{70} = \begin{cases} 1 - t(.040) & 0 \le t \le 1 \\ (.96)(1 - (t-1).044) & 1 < t < 2 \end{cases}$$

is based on ${}_tp_x = 1 - {}_tq_x = 1 - t \cdot q_x$ for $0 \le t \le 1$ under UDD. Finally

$$\overset{\circ}{e}_{70:\overline{1.5}|} = \int_0^{1.5} {}_tp_{70} \, dt = \int_0^1 {}_tp_{70} \, dt + \int_1^{1.5} {}_tp_{70} \, dt = .980 + .4747 = 1.4547,$$

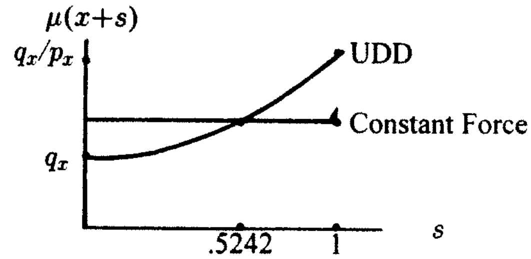
ANSWER C

ALTERNATE GEOMETRIC METHOD:
$\overset{\circ}{e}_{70:\overline{1.5}|}$ is represented as the sum of the two shaded trapezoidal regions in the figure. Using the fact that the area of a trapezoid is the length of the base times the height at the midpoint of the base we have

$$\overset{\circ}{e}_{70:\overline{1.5}|} = (1)\left(\frac{s_X(0) + s_X(1)}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{s_X(1) + s_X(1.5)}{2}\right)$$

where $s_X(0) = 1$, $s_X(1) = {}_1p_{70} = .96$ and $s_X(1.5) = \frac{s_X(1) + s_X(2)}{2} = \frac{.96 + (.96)(.956)}{2}$.

S-15. Under UDD, $\mu(x+s) = \frac{q_x}{1 - sq_x}$ for $0 \le s \le 1$ and under constant force $\mu(x+s) = -\ln p_x$ for $0 \le s \le 1$. The graphs are shown below.



So $\mu^B(x+s) \ge \mu^A(x+s)$ translates into $\frac{.25}{1 - .25s} \ge -\ln(.75)$, or $\frac{1}{4 - s} \ge .2877 = \ln\left(\frac{4}{3}\right)$. This inequality has solution set $s \ge \frac{.1508}{.2877} = .5242$, ANSWER D

S-16. Proceeding as arrows in the diagram indicate

$$\begin{array}{ccccc} \ell_{[96]} & \rightarrow & \ell_{[96]+1} & \rightarrow & \ell_{98} \\ & & & & \downarrow \\ \ell_{[97]} & \leftarrow & \ell_{[97]+1} & \leftarrow & \ell_{99} \end{array}$$

$$\ell_{[96]+1} = (1 - q_{[96]})\ell_{[96]} = \left(1 - \tfrac{1}{2}q_{96}\right)10{,}000 = \left(1 - \tfrac{1}{2}(.350)\right)10{,}000 = 8250$$

$$\ell_{98} = (1 - q_{[96]+1})\ell_{[96]+1} = \left(1 - \tfrac{1}{2}(.475)\right)(8250) = 6290.625$$

$$\ell_{99} = (1 - q_{98})\ell_{98} = (1 - .675)6290.625 = 2044.453$$

$$\ell_{[97]+1} = \ell_{99}/(1 - q_{[97]+1}) = 2044.453/\left(1 - \tfrac{1}{2}(.675)\right) = 3085.967$$

$$\ell_{[97]} = \ell_{[97]+1}/(1 - q_{[97]}) = 3085.967/\left(1 - \tfrac{1}{2}(.475)\right) = 4047.170,$$ ANSWER A

S-17. Since there is a 2-year select period, the unknown $\ell_{[32]+1}$ sits in between the given $\ell_{[32]} = 90$ and $\ell_{34} = 63$ with

$$63 = \ell_{34} = \ell_{[32]} \cdot p_{[32]} \cdot p_{[32]+1} = (90)(1 - q_{[32]})(1 - q_{[32]+1})$$

$$= (90)(1 - (1-2k)q_{32})(1 - (1-k)q_{33}).$$

$$\underset{\frac{100-90}{100}}{\uparrow} \qquad \underset{\frac{90-63}{90}}{\uparrow}$$

This tedious quadratic equation in k has the solution $k = \frac{1}{6}$ so that

$$\ell_{[32]+1} = \ell_{[32]} \cdot p_{[32]} = 90\left(1 - \left(1-2 \cdot \tfrac{1}{6}\right)\tfrac{10}{100}\right) = 84,$$ ANSWER C

S-18. In general $\overset{\circ}{e}_{x:\overline{1}|} = \int_0^1 {}_tp_x dt$. Under the UDD assumption ${}_tp_x = 1 - tq_x = 1 - .1t$, which results in

$$F = \overset{\circ}{e}_{x:\overline{1}|} = \int_0^1 (1 - .1t)\,dt = 1 - .05 = .95.$$

Balducci's law, linear interpolation with reciprocals, gives a bit more complicated expression for ${}_tp_x = \ell_{x+t}/\ell_x$:

$$ {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \left(\frac{\ell_x}{\ell_{x+t}}\right)^{-1} = \left[\ell_x\left(\frac{1-t}{\ell_x} + \frac{t}{\ell_{x+1}}\right)\right]^{-1}$$

$$= \left[(1-t) + \frac{t}{p_x}\right]^{-1} = \left[1 - t + \tfrac{10}{9}t\right]^{-1} = 9[9+t]^{-1}$$

$$G = \overset{\circ}{e}_{x:\overline{1}|} = \int_0^1 9(9+t)^{-1}dt = 9\,ln(9+t)\Big|_0^1 = 9(ln\,10 - ln\,9) = .94825.$$

Hence $1000(F - G) = 1.755$, ANSWER D.

Note: See the example under the heading "Fractional Age Assumptions" in the Condensed Review Notes for Unit 1.

S-19. With $s_X(x) = \left(1 - \frac{x}{\omega}\right)^r$ it follows that

$$.10 = \mu(y) = \frac{-s_Y'(y)}{s_Y(y)} = \frac{r}{\omega}\left(1 - \frac{y}{\omega}\right)^{r-1}\Big/\left(1 - \frac{y}{\omega}\right)^r = \frac{r}{\omega - y}$$

$$8.75 = \overset{\circ}{e}_y = \int_0^{\omega - y} {}_tp_y\,dt = \int_0^{\omega-y} \frac{s_Y(t+y)}{s_Y(y)}\,dt = \int_0^{\omega-y} \frac{(\omega - y - t)^r}{(\omega - y)^r}\,dt = \frac{\omega - y}{r+1}.$$

Notice then that $(.10)(8.75) = \frac{r}{\omega - y}\cdot\frac{\omega - y}{r+1} = \frac{r}{r+1}$, which results in $r = 7$, ANSWER D.

Note: See the third example under the heading "Life Expectancy" in the Condensed Review Notes for Unit 1.

S-20. With $q_x = .12$ we can use $\ell_x = 100$, $\ell_{x+1} = 88$.

I.
$$ {}_{1/3}q_{x+1/2} = \frac{\ell_{x+1/2} - \ell_{x+5/6}}{\ell_{x+1/2}} \underset{\text{UDD}}{=} \frac{\left(\ell_x - \frac{1}{2}d_x\right) - \left(\ell_x - \frac{5}{6}d_x\right)}{\ell_x - \frac{1}{2}d_x}$$

$$= \frac{\frac{1}{3}d_x}{\ell_x - \frac{1}{2}d_x} = \frac{4}{94} = .042553 = .0426$$

So I is true (presuming "=" means "equal to 4 places")

II.
$$ {}_{1/3}q_x = \frac{\ell_x - \ell_{x+1/3}}{\ell_x}$$

Under Balducci's law $\frac{1}{\ell_{x+1/3}} = \frac{2}{3}\frac{1}{\ell_x} + \frac{1}{3}\frac{1}{\ell_{x+1}} = \frac{2}{300} + \frac{1}{264}$, so $\ell_{x+1/3} = 95.652174$ and ${}_{1/3}q_x = .043478 = .0435$. II is also true.

III. ${}_{1/2}q_x = 1 - {}_{1/2}p_x$, and under constant force ${}_tp_x = e^{-\mu t} = (e^{-\mu})^t = p_x^t$. Hence ${}_{1/2}p_x = (.88)^{1/2}$ and ${}_{1/2}q_x = .061917 = .0619$. III is correct, ANSWER D

S-21. The data in (iii) alone is sufficient to find q_y using

$$q_y = 1 - p_y = 1 - e^{\int_0^1 \mu(y+t)dt}.$$

A short cut here results from observing that $\mu(y+t)$ is exactly double $\mu(x+t)$. Hence

$$p_y = e^{\int_0^1 \mu(y+t)dt} = e^{\int_0^1 2\mu(x+t)dt} = \left[e^{\int_0^1 \mu(x+t)dt}\right]^2 = p_x^2.$$

Thus

$$q_y = 1 - p_y = 1 - p_x^2 = 1 - (1-q_x)^2$$
$$= 1 - .96^2 = .0784.$$

ANSWER A

S-22. I.

$${}_2p_{[31]} = \frac{988}{996} \text{ (bigger)}, \quad {}_2p_{[30]+1} = \frac{988}{998} \quad \Rightarrow \text{ true}$$

II.

$${}_{1|}q_{[31]} = \frac{994-988}{996} = \frac{6}{996}, \quad {}_{1|}q_{[30]+1} = \frac{995-988}{998} = \frac{7}{998} \text{ (bigger)} \quad \Rightarrow \text{ false}$$

III.

$${}_2q_{[33]} = \frac{987-970}{987} = \frac{17}{987}, \quad {}_2q_{[31]+2} = \frac{988-970}{988} = \frac{18}{988} \text{ (bigger)} \quad \Rightarrow \text{ false}$$

ANSWER B

S-23. Under de Moivre's law $\ell_x = \omega - x$ and $T(x)$ is uniformly distributed over $[0, \omega - x]$. Hence

$$\mu(x) = \frac{-\ell_x'}{\ell_x} = \frac{1}{\omega - x}, \quad \overset{\circ}{e}_x = E[T(x)] = \frac{\omega - x}{2}.$$

$${}_{n|}q_x = \frac{\ell_{x+n} - \ell_{x+n+1}}{\ell_x} = \frac{1}{\omega - x}, \quad L_x = \tfrac{1}{2}\ell_x + \tfrac{1}{2}\ell_{x+1} \text{ (UDD)} = \omega - x - \tfrac{1}{2}$$

$$m_x = \frac{d_x}{L_x} = \frac{1}{\omega - x - .5}.$$

I. $\dfrac{1}{2\overset{\circ}{e}_x} = \dfrac{1}{2(\omega - x)/2} = \dfrac{1}{\omega - x}$ correct

II. ${}_{n|}q_x = \dfrac{1}{\omega - x}$ correct

III. $\dfrac{m_x}{1 + .5m_x} = \dfrac{2/(2\omega - 2x - 1)}{2(\omega - x)/(2\omega - 2x - 1)} = \dfrac{1}{\omega - x}$ correct

ANSWER D

S-24. The basic properties of a survival function are: $s_X(0) = 1$, it must be non-increasing, and $\lim_{x \to \infty} s_X(x) = 0$.

I. $s_X(0) = e^0 = 1$

$$s'_X(x) = exp(x - .7(2^x - 1)) \underbrace{(x - .7(2^x - 1))'}_{1 - .7(2^x) ln2 = 1 - (.4852)2^x}$$

$\Rightarrow \quad s'_X(0) = .5148 \quad \Rightarrow$ increasing near 0

$\therefore$ not a survival function.

II.

$$\left.\begin{array}{l} s_X(0) = \dfrac{1}{(1+0)^2} = 1 \\ s'_X(x) = -2(1+x)^{-3} < 0 \Rightarrow \text{decreasing} \\ \lim_{x\to\infty} s_X(x) = \lim_{x \to \infty} \dfrac{1}{(1+x)^2} = \dfrac{1}{\infty} = 0 \end{array}\right\} s_X(x) \text{ is a survival function.}$$

III.

$$\left.\begin{array}{l} s_X(0) = e^0 = 1 \\ s'_X(x) = (e^{-x^2})(-2x) < 0 \Rightarrow \text{decreasing} \\ \lim_{x\to\infty} s_X(x) = e^{-\infty} = 0 \end{array}\right\} s_X(x) \text{ is a survival function.}$$

ANSWER C

S-25. The problem gives a discrete distribution (that of K), whereas $\mathring{e}_{x:\overline{5}|}$ is the expected value of a function of the continuous variable T. UDD is a bridge between these two situations, giving $\mathring{e}_{x:\overline{5}|} = e_{x:\overline{5}|} + \frac{1}{2} \cdot {}_5q_x = \sum_{k=1}^{5} {}_kp_x + \frac{1}{2} \cdot {}_5q_x$, by the standard UDD relation.

Now $F_K(k) = Pr(K \leq k) = Pr(T < k+1) = {}_{k+1}q_x$, so ${}_{k+1}p_x = 1 - F_K(k)$, and therefore

$\mathring{e}_{x:\overline{5}|} = .95 + .85 + .65 + .40 + .25 + \left(\frac{1}{2}\right)(.75) = 3.475$, ANSWER B

S-26. Working from first principles $\mu(x) = \dfrac{-s'_X(x)}{s_X(x)} = \dfrac{\alpha}{\omega - x}$,

$$\mathring{e}_x = \int_0^{\omega - x} {}_tp_x \, dt = \int_0^{\omega - x} \frac{s_X(x+t)}{s_X(x)} dt = \int_0^{\omega - x} \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} dt = \frac{\omega - x}{\alpha + 1}.$$

Thus $\mu(x) \cdot \mathring{e}_x = \left(\frac{\alpha}{\omega - x}\right)\left(\frac{\omega - x}{\alpha + 1}\right) = \frac{\alpha}{\alpha + 1}$. ANSWER A

Note: The model $\mu(x) = \frac{r}{\omega - x}$ has frequently occurred on past exams and several shortcuts can be learned. See the examples in the Unit Condensed Review Notes.

S-27. The probability that 60 dies between 60.5 and 61.5, ${}_{.5|}q_{60}$, can be calculated from $\frac{\ell_{60.5} - \ell_{61.5}}{\ell_{60}}$. The mid-year values $\ell_{60.5}$, $\ell_{61.5}$ are calculated here with an interpolation assumption from ℓ_{60}, ℓ_{61} and ℓ_{62}.

From $q_{60} = .30$, $q_{61} = .40$ it is easy to see that appropriate ℓ_x values are $\ell_{60} = 100$, $\ell_{61} = 70$, $\ell_{62} = 42$.

UDD: $\ell_{60.5} = \frac{100+70}{2} = 85$, $\ell_{61.5} = \frac{70+42}{2} = 56$, $f = {}_{.5|}q_{60}^{UDD} = \frac{85-56}{100} = .29000$

Balducci: $\frac{1}{\ell_{60.5}} = \frac{1}{2} \cdot \frac{1}{100} + \frac{1}{2} \cdot \frac{1}{70} \Rightarrow \ell_{60.5} = 82.3529$. Similarly $\ell_{61.5} = 52.5000$, so

$$g = {}_{.5|}q_{60}^{Bal} = \frac{82.3529 - 52.5000}{100} = .29853$$

$$\Rightarrow \quad 10{,}000(g - f) = 85.3.$$

ANSWER B

S-28. The median sought, $t_{.5}$, is the solution of $.5 = S(t_{.5}) = {}_{t_{.5}}p_{20} = exp\left(-\int_0^{t_{.5}} \mu(20+t)dt\right)$:

$-\int_0^{t_{.5}} \frac{1}{\sqrt{60-s}}\, ds = 2(\sqrt{60-t_{.5}} - \sqrt{60})$. Substituting into the above and taking a natural log of both sides results in $-ln2 = 2(\sqrt{60-t_{.5}} - \sqrt{60})$: $t_{.5} = 5.249$

ANSWER A

S-29. The key to this problem is recognizing that ${}_tp_x = .8^t$ assumes constant force with $e^{-\mu t} = .8^t$, $\mu = -ln(.8)$. Now $T_{x+1} = \int_{x+1}^{\infty} \ell_y\, dy$ is used in calculating $\mathring{e}_{x+1} = T_{x+1}/\ell_{x+1}$. Under a constant force assumption both $T(x)$ and $T(x+1)$ have the same exponential distribution. Hence $\frac{-1}{ln(.8)} = \frac{1}{\mu} = E[T(x+1)] = \mathring{e}_{x+1} = \frac{T_{x+1}}{\ell_{x+1}}$. Finally, $\ell_{x+2} = \ell_{x+1} \cdot p_{x+1}$ gives $\ell_{x+1} = 6.4/.8 = 8$ and $T_{x+1} = \frac{-8}{ln(.8)} = 35.85$.

ANSWER D

S-30. A natural way to set this one up, due to the 10-year select period, is

$$\mathring{e}_{[30]} = \underbrace{\mathring{e}_{[30]:\overline{10}|}}_{\text{based on select mortality}} + {}_{10}p_{[30]} \cdot \underbrace{\mathring{e}_{40}}_{\text{ultimate mortality}}$$

(See the Unit Condensed Review Notes under the heading "Recursion Relations".) Now ${}_tp_{[30]} = \frac{\ell_{[30]+t}}{\ell_{[30]}} = 1 - \frac{t}{100}$ for $0 \le t \le 10$ so $\mathring{e}_{[30]:\overline{10}|} = \int_0^{10}\left(1 - \frac{t}{100}\right)dt = 9.5$. ${}_{10}p_{[30]} = 1 - \frac{10}{100} = .90$. Next, ${}_tp_{40} = \frac{\ell_{40+t}}{\ell_{40}} = \frac{\ell_{30+(10+t)}}{\ell_{30+10}} = \sqrt{\frac{60-t}{60}} = \left(1 - \frac{t}{60}\right)^{1/2}$, hence

$\mathring{e}_{40} = \int_0^{60}\left(1 - \frac{t}{60}\right)^{1/2} dt = \frac{60}{1.5} = 40$. Substituting these results into the first relation results in $\mathring{e}_{[30]} = 9.5 + (.90)40 = 45.5$.

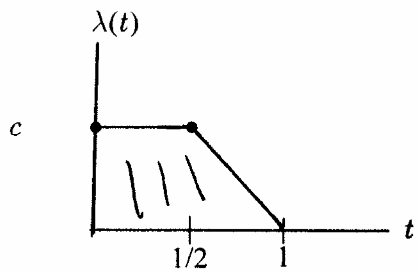
ANSWER D

S-31. This question concerns the effect on q_x of an additional hazard rate of $\lambda(t)$ at age $x+t$ for $0 \le t \le 1$:

$$\underbrace{q'_x}_{\text{with additional hazard}} = 1 - p'_x = 1 - e^{-\int_0^1 \mu(x+t)+\lambda(t)\,dt}$$

$$= 1 - \underbrace{(e^{-\int_0^1 \mu(x+t)\,dt})}_{p_x}\underbrace{(e^{-\int_0^1 \lambda(t)\,dt})}_{\text{factor } < 1}.$$

The definition of $\lambda(t)$ in this problem is given graphically by



From simple geometry the shaded area is $\int_0^1 \lambda(t)\,dt = .75c$. Thus
$q'_x = 1 - p_x e^{-.75c} = 1 - (.985)e^{-.75c}$. ANSWER E

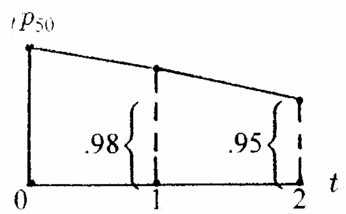
S-32. The question can be done from first principles or from recognition that the survival model in the problem is a slightly disguised form of $s_X(x) = \left(1 - \frac{x}{\omega}\right)^r$, discussed in the examples of the unit Condensed Review Notes. Here $s_X(x) = \frac{\sqrt{k^2 - x}}{k} = \left(1 - \frac{x}{k}\right)^{1/2}$ so $\mathring{e}_x = \frac{\omega - x}{r+1} = \frac{k-x}{1.5}$. From $\mathring{e}_{40} = 2\mathring{e}_{80}$ it follows that $\omega = k = 120$, hence $\mathring{e}_{60} = \frac{\omega - 60}{1.5} = \frac{60}{1.5} = 40$. ANSWER D

S-33-36. A useful point in this problem is the evaluation of $\int_0^n {}_tp_x\,dt$ when you are given a tabular survival model and told to assume UDD.

EXAMPLE:

x	50	51	52
ℓ_x	100	98	95

Under the UDD the survival function ${}_tp_{50}$, $0 \le t \le 2$ is piecewise linear as illustrated in the graph



So $\int_0^2 {}_tp_{50}\,dt$ is the sum of two trapezoidal areas given by

$$\int_0^2 {}_tp_{50}\,dt = \tfrac{1}{2}(1+.98) + \tfrac{1}{2}(.98+.95) = \tfrac{1}{2}\cdot 1 + (.98) + \tfrac{1}{2}\cdot(.95) = 1.955$$

We are given $\ell_{[85]} = 1000$, $\ell_{86} = 1000 - 100 = 900$, $\ell_{[86]} = 850$ and $\ell_{87} = 850 - 100 = 750$ since the select period is 1 year. Hence

$$p_{[85]} = \frac{900}{1000} = .9,\ \ p_{[86]} = \frac{750}{850},\ \ p_{86} = \frac{750}{900}.$$

Question 33 has ANSWER C.

For this part the discussion above is relevant:

$$5.225 = \overset{\circ}{e}_{[85]} = \int_0^\infty {}_tp_{[85]}\,dt = \int_0^2 {}_tp_{[85]}\,dt + \int_2^\infty {}_tp_{[85]}\,dt.$$

By above

$$\int_0^2 {}_tp_{[85]}\,dt = \tfrac{1}{2}\cdot 1 + p_{[85]} + \tfrac{1}{2}\cdot {}_2p_{[85]}$$

$$= .50 + .90 + \tfrac{1}{2}\cdot p_{[85]}\cdot p_{86} = .50 + .90 + \tfrac{1}{2}(.75) = 1.775,$$

so $5.225 = 1.775 + \int_2^\infty {}_tp_{[85]}\,dt$; $\int_2^\infty {}_tp_{[85]}\,dt = 3.45.$

Question 34 has ANSWER C.

From standard life-expectancy recursion relations (see the end of the unit Condensed Review Notes)

$$\overset{\circ}{e}_{[86]} = \int_0^1 {}_tp_{[86]}\,dt + p_{[86]}\cdot \overset{\circ}{e}_{87} = \tfrac{1}{2}(1+p_{[86]}) + p_{[86]}\cdot \overset{\circ}{e}_{87} = \frac{800}{850} + \frac{750}{850}\cdot \overset{\circ}{e}_{87}$$

Going back to the above we had $5.225 = \overset{\circ}{e}_{[85]} = \overset{\circ}{e}_{[85]:\overline{2}|} + {}_2p_{[85]}\cdot \overset{\circ}{e}_{87} = 1.775 + 3.45$ so $\overset{\circ}{e}_{87} = \frac{3.45}{{}_2p_{[85]}} = \frac{3.45}{.75} = 4.60$. Hence $\overset{\circ}{e}_{[86]} = \frac{800}{850} + \frac{750}{850}(4.60) = 5.$

So Questions 35 and 36 are both ANSWER E.

S-37-40 $s_X(x) = \frac{(10-x)^2}{100}$ is of the form $\left(1-\frac{x}{\omega}\right)^r$, for which the following can be written down from memory (see the examples in the Unit Condensed Review Notes): $\overset{\circ}{e}_x = \frac{\omega - x}{r+1}$, $\mu(x) = \frac{r}{\omega - x}$, $\ell_x = (\omega - x)^r$.

For Question 37 we use $\overset{\circ}{e}_1 = \frac{\omega - 1}{r+1} = \frac{10-1}{3} = 3$. ANSWER E.

For Question 38 we use $\mu(1) = \frac{r}{\omega - 1} = \frac{2}{10-1} = \frac{2}{9}$ and $q_1 = 1 - \frac{\ell_2}{\ell_1} = 1 - \frac{8^2}{9^2} = \frac{17}{81}$. Thus $\mu(1) - q_1 = \frac{2}{9} - \frac{17}{81} = \frac{1}{81} = .012$. ANSWER C.

For the final two questions we are asked to calculate $a(1)$. By a standard method (see Unit Condensed Review Notes)

$\overset{\circ}{e}_{1:\overline{1}|} = q_1 \cdot a(1) + p_1 \cdot 1$.

Here, using $q_1 = \frac{17}{81}$, $p_1 = \frac{64}{81}$, we have $a(1) = \frac{\overset{\circ}{e}_{1:\overline{1}|} - (64/81)}{(17/81)}$

The final two questions require calculations of $\overset{\circ}{e}_{1:\overline{1}|}$ (hence $a(1)$) under slightly different assumptions and result in slightly different values of $a(1)$.

Question 39 calculation of $\overset{\circ}{e}_{1:\overline{1}|}$ and $a(1)$:

$$\overset{\circ}{e}_{1:\overline{1}|} = \int_0^1 {}_tp_1\, dt = \int_0^1 S(1+t)/S(1) dt = \int_0^1 \left(1 - \frac{t}{9}\right)^2 dt = \frac{9^3 - 8^3}{243} = .8930$$

$$a(1) = \frac{.8930 - (64/81)}{(17/81)} = .490, \text{ ANSWER D.}$$

Question 40 calculation of $\overset{\circ}{e}_{1:\overline{1}|}$ and $a(1)$:
Here we use ${}_tp_1 = p_1^t$ (constant force assumption in place of the actual $S(1+t)/S(1)$) to calculate

$$\overset{\circ}{e}_{1:\overline{1}|} = \int_0^1 p_1^t\, dt = \frac{-q_1}{\ln p_1} = .8909$$

$$a(1) = \frac{.8909 - (64/81)}{(17/81)} = .480, \text{ ANSWER B.}$$

S-41-44. The given T-density is exponential with parameter $\mu = 2$. From standard results the answer to Question 41 is $\overset{\circ}{e}_x = E[T] = \frac{1}{\mu} = .5$, ANSWER A, and the answer to Question 42 is $Var(T) = \frac{1}{\mu^2} = .25$, ANSWER A.

The survival function $S_T(t) = {}_tp_x = e^{-\mu t} = e^{-2t}$ reaches a value of .5 at the median time $t = \frac{\ln(2)}{\mu} = \frac{\ln(2)}{2}$. Question 43 has ANSWER C.

Finally, since a central rate is a weighted average of the force over an age interval, an assumption of constant force means $m_x = \mu = 2$. Question 44 has ANSWER E.