LIFE CONTINGENCIES

Single Life Model

Basic Random Variables
X – denotes the age-at-death random variable

(x) – denotes a life aged x
(ie someone who's already survived to age x)

ω – denotes terminal age (unless otherwise stated, we assume $\omega = \infty$)
(no one survive past age ω)

$T = T(x)$ – denotes the continuous future lifetime of (x) random variable
$T = X - x \mid X > x$
Note that $X = T(0)$

$K = K(x)$ – denotes the curtate future lifetime of (x) random variable
This is a discretization of the T random variable.

The Age-at-Death Random Variable: *X*

$Supp(X) = (0, \omega)$

(cumulative distribution function – cdf)

$F_X(x) = \Pr(X \le x) = {}_x q_0$ (Recall that $F' = f$, the probability density function (pdf))

(survival function – sf)

$s_X(x) = \Pr(X > x) = {}_x p_0 = 1 - {}_x q_0$ (Note that $s' = -f$)

(force of mortality – fom) aka hazard rate or failure rate

$$\mu(x) = \frac{f_X(x)}{s_X(x)} = -\frac{d}{dx}\left[\ln(s_X(x)\right]$$

Comments and Concepts

1. The force of mortality is the rate of mortality at a particular point in time. The expression $\mu(x)dx$ represents the probability that a newborn that has survived to age *x* dies in the next instant "*dx*".

2. ${}_x p_0 = \exp(-\int_0^x \mu(s)ds)$

3. ${}_x q_0 = \Pr(X \le x) = \int_0^x f_X(s)ds = \int_0^x {}_s p_0 \mu(s)ds$

The (continuous) Future Lifetime Random Variable: *T* = *T*(*x*)
$Supp(T) = (0, \omega - x)$

(pdf): $f_T(t) = \dfrac{f_X(x+t)}{\Pr(X > t)}$

(cdf): $F_T(t) = \Pr(T \le t) = {}_t q_x = \dfrac{\Pr(x < X \le x+t)}{\Pr(X > x)} = 1 - \dfrac{\Pr(X > x+t)}{\Pr(X > x)}$

(sf): $s_T(t) = \Pr(T > t) = {}_t p_x = \dfrac{\Pr(X > x+t)}{\Pr(X > x)} = \dfrac{{}_{x+t}p_0}{{}_x p_0} = 1 - {}_t q_x$

Comments and Concepts

1. $f_T(t) = {}_t p_x \cdot \mu(x+t)$

2. ${}_t q_x = \int_0^t {}_s p_x \cdot \mu(x+s)ds$

3. ${}_t p_x = \exp(-\int_0^t \mu(x+s)ds) = \exp(-\int_x^{x+t} \mu(s)ds$ Also, ${}_t p_x = \int_t^\infty {}_s p_x \cdot \mu(x+s)ds$

4. ${}_1 q_x = q_x$ and ${}_1 p_x = p_x$

5. (survival factorization) ${}_{n+t} p_x = {}_n p_x \cdot {}_t p_{x+n}$

6. (deferred mortality)
$${}_{t|u} q_x = \Pr(t < T(x) \le t+u) = \int_t^{t+u} {}_s p_x \cdot \mu(x+s)ds = {}_t p_x - {}_{t+u} p_x = {}_{t+u} q_x - {}_t q_x = {}_t p_x \cdot {}_u q_{x+t}$$

7. (complete expectation of life for (*x*), aka mean residual lifetime)
$\overset{0}{e}_x = E[T(x)] = \int_0^\infty t \cdot {}_t p_x \cdot \mu(x+t)dt = \int_0^\infty {}_t p_x dt$ (The last equality is used often.)

8. (n-year temporary complete expectation of life for (x); this is the expected number of years lived by (x) between ages x and $x+n$)

$$\overset{0}{e}_{x:\overline{n|}} = E[T(x) \wedge n] = \int_0^n t \cdot {}_t p_x \cdot \mu(x+t)dt + n \cdot {}_n p_x = \int_0^n {}_t p_x dt$$

9. (Recursion Formulas)

$\overset{0}{e}_x = \overset{0}{e}_{x:\overline{n|}} + {}_n p_x \cdot \overset{0}{e}_{x+n}$ and when $n = 1$ we get $\overset{0}{e}_x = \int_0^1 {}_t p_x dt + p_x \cdot \overset{0}{e}_{x+1}$

$$\overset{0}{e}_{x:\overline{n|}} = \overset{0}{e}_{x:\overline{m|}} + {}_m p_x \cdot \overset{0}{e}_{x+m:\overline{n-m|}}$$

The Curtate Future Lifetime Random Variable: *K* = *K*(*x*)
$Supp(K) = (0,1,2,\cdots,\omega - x - 1)$

The probability distribution table for *K* is

K	Pr
0	$\Pr(0 \le T < 1) = \Pr(0 < T \le 1) = q_x (= {}_{0|}q_x)$
1	$\Pr(1 \le T < 2) = \Pr(1 < T \le 2) = {}_{1|}q_x$
2	$\Pr(2 \le T < 3) = \Pr(2 < T \le 3) = {}_{2|}q_x$
$\vdots$	$\vdots$

Note: ${}_{n|}q_x = {}_np_x - {}_{n+1}p_x = {}_{n+1}q_x - {}_nq_x = {}_np_x \cdot q_{x+n}$

(cdf): $F_K(k) = \Pr(K \le k) = \sum_{n=0}^{k} {}_{n|}q_x$ (Notice that $\Pr(K \le k) = \Pr(T \le k+1) = {}_{k+1}q_x$)

Comments and Concepts

1. (curtate expectation of life for (*x*))
   $e_x = E[K(x)] = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x$
   (The last equality is used often. Notice the index starts at $k = 1$.)

2. $E[(K(x))^2] = \sum_{k=0}^{\infty} k^2 \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} (2k-1) \cdot {}_kp_x$ (Helps us get variance of *K*.)

3. (n-year temporary curtate expectation of life for (*x*); this is the expected complete number of years lived by (*x*) between ages *x* and *x*+*n*)
   $e_{x:\overline{n|}} = \sum_{k=1}^{n} {}_kp_x$

4. (Recursion Formulas)
   $e_x = e_{x:\overline{n|}} + {}_np_x \cdot e_{x+n}$ and when $n = 1$ we get $e_x = p_x \cdot (1 + e_{x+1})$
   $e_{x:\overline{n|}} = e_{x:\overline{m|}} + {}_mp_x \cdot e_{x+m:\overline{n-m|}}$

Life Table Notation

l_0 = (arbitrary) number of newborns

$l_x = l_0 \cdot {}_x p_0$ = (expected) number of survivors at age x (Note: ${}_x p_0 = \frac{l_x}{l_0}$)

${}_n d_x = l_x - l_{x+n} =$ number of deaths between ages x and $x+n$

($ {}_1 d_x = d_x$)

Note: ${}_n d_x = d_x + d_{x+1} + \cdots + d_{x+n-1}$

Formulas and concepts involving life table notation

1. ${}_n p_x = \frac{l_{x+n}}{l_x}$ and ${}_n q_x = \frac{{}_n d_x}{l_x} = \frac{l_x - l_{x+n}}{l_x} = 1 - {}_n p_x$ and ${}_{n|m} q_x = \frac{{}_m d_{x+n}}{l_x} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$

2. $\mu(x) = \frac{-\frac{d}{dx}[l_x]}{l_x}$, and so $\frac{d}{dx}[l_x] = -l_x \cdot \mu(x)$

3. ${}_n d_x = \int_0^n l_{x+t} \cdot \mu(x+t)dt$ (follows since ${}_n q_x = \int_0^n {}_t p_x \cdot \mu(x+t)dt$)

4. ${}_n L_x =$ the total number of years lived in the next n years by the l_x people alive at age x

$${}_n L_x = \int_0^n l_{x+t}dt = \int_x^{x+n} l_y dy = n \cdot l_{x+n} + {}_n d_x \cdot E[T(x) | T(x) < n]$$

Note that $E[T(x) | T(x) < n] = \frac{\int_0^n t \cdot {}_t p_x \cdot \mu(x+t)dt}{{}_n q_x} = \frac{\int_0^n t \cdot l_{x+t} \cdot \mu(x+t)dt}{{}_n d_x}$

5. $\frac{d}{dx}[{}_n L_x] = l_{x+n} - l_x = -{}_n d_x$

Life Table Notation (continued)

6. $T_x = {}_{\infty}L_x$ the total number of years lived in the future by the l_x people alive at age x

7. ${}_nL_x = T_x - T_{x+n}$ and $\frac{d}{dx}[T_x] = -l_x$

8. $\overset{0}{e}_x = \frac{T_x}{l_x}$ and $\overset{0}{e}_{x:\overline{n}|} = \frac{{}_nL_x}{l_x}$

9. $E[(T(x))^2] = \frac{2Y_x}{l_x}$ where $Y_x = \int_0^{\infty} T_{x+t}\,dt$

10. ${}_nm_x = \frac{{}_nd_x}{{}_nL_x}$ is the n-year central mortality weight

Extending Discrete to Continuous (Fractional Age Assumptions)
(x is an integer and all formulas are valid for $0 \le t \le 1$)

UDD (Uniform Distribution of Deaths: $_t d_x = t \cdot d_x$)

$_t q_x = t \cdot q_x$, and so $_t p_x = 1 - t \cdot q_x$, and $f_T(t) = q_x$ (a constant wrt t)

Then $l_{x+t} = l_x - t \cdot d_x$ and $\mu(x+t) = \dfrac{q_x}{1 - t \cdot q_x}$ and $m_x = \dfrac{q_x}{1 - 0.5q_x}$

Defining $V(x)$ to be the random variable representing the fraction of the year lived in the year in which (x) dies, we can relate the random variables T and K; namely, $T(x) = K(x) + V(x)$, and K and V are independent.

Under UDD, we have $V(x) \sim U(0,1)$.
Then $\overset{0}{e}_x = e_x + 0.5$. Also $\overset{0}{e}_{x:\overline{n|}} = e_{x:\overline{n|}} + 0.5({}_n q_x)$ and $Var(T(x)) = Var(K(x)) + \dfrac{1}{12}$

For $0 \le s + t \le 1$, $_t q_{x+s} = \dfrac{t \cdot q_x}{1 - s \cdot q_x}$ and $_t p_{x+s} \cdot \mu(x+s+t) = \dfrac{q_x}{1 - s \cdot q_x}$

Constant Force (Exponential Interpolation: $\mu(x+t) = \mu$)

For $0 \le s + t \le 1$, $_t p_{x+s} = e^{-\mu \cdot t} = (p_x)^t$

$m_x = \mu$

Balducci (Hyperbolic Interpolation)

$$\frac{1}{l_{x+t}} = \frac{1}{l_x} + t\left(\frac{1}{l_{x+1}} - \frac{1}{l_x}\right)$$

$\mu(x+t) = \dfrac{q_x}{1 - (1-t) \cdot q_x}$ and for $0 \le s + t \le 1$, $_t q_{x+s} = \dfrac{t \cdot q_x}{1 - (1 - s - t) \cdot q_x} = t \cdot \mu(x+s+t)$

Select and Ultimate Mortality

Select mortality rates are used for a period (usually 3 years or less) and are different than the ultimate (general population) rates.

$[x]$ denotes an x-year old for which select rates are used starting at age x

$[x] + k$ denotes an $(x + k)$-year old for which select rates are used starting at age x

$[x + k]$ denotes an $(x + k)$-year old for which select rates are used starting at age $x + k$

$(x + k)$ denotes an $(x + k)$-year old for which ultimate rates are used starting at age $x + k$

Comments:

1. $[x] + k = (x + k)$ if k exceeds the select period

2. The force of mortality for $[x]$ is denoted $\mu_x(t)$.

Multiple-Life Models

For the moment, we consider only two *independent* lives (x) and (y). The models can easily be extended to more than two lives.

Joint-Life Status: $T(xy) = Min\{T(x), T(y)\}$
(Joint-life status fails on the earlier of the deaths of (x) and (y))

$$ {}_t p_{xy} = \Pr(T(xy) > t) = \Pr(T(x) > t) \cdot \Pr(T(y) > t) = {}_t p_x \cdot {}_t p_y $$

$$ {}_t q_{xy} = 1 - {}_t p_{xy} $$

$$ \overset{0}{e}_{xy} = \int_0^\infty {}_t p_{xy} \, dt $$

$$ e_{xy} = \sum_{k=1}^{\infty} {}_k p_{xy} $$

$$ {}_{t|u} q_{xy} = {}_t p_{xy} - {}_{t+u} p_{xy} $$

$$ \mu_{xy}(t) = \mu_x(t) + \mu_y(t) $$

Last-Survivor Status: $T(\overline{xy}) = Max\{T(x), T(y)\}$
(Last-survivor status fails on the latest of the deaths of (x) and (y))

For independent lives (x) and (y):

$$ {}_t q_{\overline{xy}} = \Pr(T(\overline{xy}) \le t) = \Pr(T(x) \le t) \cdot \Pr(T(y) \le t) = {}_t q_x \cdot {}_t q_y $$

$$ {}_t p_{\overline{xy}} = 1 - {}_t q_{\overline{xy}} $$

$$ {}_{t|u} q_{\overline{xy}} = {}_{t+u} q_{\overline{xy}} - {}_t q_{\overline{xy}} $$

$$ \mu_{\overline{xy}}(t) = \frac{{}_t p_x \cdot \mu_x(t) + {}_t p_y \cdot \mu_y(t) - {}_t p_{xy} \cdot \mu_{xy}(t)}{{}_t p_{\overline{xy}}} $$

Contingent Probabilities

Notation:

$_{n}q_{\overset{1}{x}y}$ = Pr((*x*) dies first and within the next *n* years)

$_{\infty}q_{\overset{1}{x}y}$ = Pr((*x*) dies first)

$_{n}q_{x\overset{2}{y}}$ = Pr((*y*) dies second and within the next *n* years)

The notation $\overset{1}{x}y$ indicates that the joint-life status (*xy*) fails due to the failure of the (*x*) status, ie the death of (*x*).

Contingent Probability Formulas:

$$_{n}q_{\overset{1}{x}y} = \int_0^n {}_{t}p_{xy} \cdot \mu_x(t)dt$$

$$_{n}q_{x\overset{2}{y}} = \int_0^n {}_{t}q_x \cdot {}_{t}p_y \cdot \mu_y(t)dt$$

Contingent Probability Relationships:

1. $_{n}q_x = {}_{n}q_{\overset{1}{x}y} + {}_{n}q_{\overset{2}{x}y}$

2. $_{n}q_y = {}_{n}q_{x\overset{1}{y}} + {}_{n}q_{x\overset{2}{y}}$

3. $_{n}q_{xy} = {}_{n}q_{\overset{1}{x}y} + {}_{n}q_{x\overset{1}{y}}$

4. $_{n}q_{\overline{xy}} = {}_{n}q_{\overset{2}{x}y} + {}_{n}q_{x\overset{2}{y}}$

5. $_{n}q_{\overset{1}{x}y} - {}_{n}q_{x\overset{2}{y}} = {}_{n}q_x \cdot {}_{n}p_y$

Common Shock Model
(In recent exams, this has been the only model tested in which the lifetimes of (*x*) and (*y*) are dependent.)

Notation: *Z* = the common shock random variable.
Always $Z \sim EX(mean = \frac{1}{\lambda})$. ($\lambda$ = the fom of the shock)

*T**(*x*) = future lifetime of (*x*) random variable in the absence of the common shock ($\Pr(T^*(x) > t) = {}_t p_x^*$)

*T**(*y*) = future lifetime of (*y*) random variable in the absence of the common shock ($\Pr(T^*(y) > t) = {}_t p_y^*$)

*Z, T**(*x*), and *T**(*y*) are all assumed mutually independent

Define $T(x) = Min\{T^*(x), Z\}$ and define $T(y) = Min\{T^*(y), Z\}$.
(*T*(*x*) and *T*(*y*) are not independent, and $T(xy) = Min\{T^*(x), T^*(y), Z\}$)

Formulas:

$${}_t p_x = \Pr(T(x) > t) = {}_t p_x^* \cdot \Pr(Z > t) = {}_t p_x^* \cdot e^{-\lambda t}$$
$${}_t p_y = \Pr(T(y) > t) = {}_t p_y^* \cdot \Pr(Z > t) = {}_t p_y^* \cdot e^{-\lambda t}$$
$${}_t p_{xy} = \Pr(T(xy) > t) = {}_t p_x^* \cdot {}_t p_y^* \cdot \Pr(Z > t) = {}_t p_x^* \cdot {}_t p_y^* \cdot e^{-\lambda t}$$

Often Tested Special Case of Common Shock Model
(*x*) and (*y*) have constant forces, μ_x and μ_y, respectively

Given: $T^*(x) \sim EX(mean = \frac{1}{\mu_x})$ and $T^*(y) \sim EX(mean = \frac{1}{\mu_y})$

Formulas:

$${}_t p_x^* = e^{-\mu_x \cdot t} \quad \text{and} \quad {}_t p_y^* = e^{-\mu_y \cdot t}$$
$${}_t p_x = e^{-(\mu_x + \lambda)\cdot t} \quad \text{and} \quad {}_t p_y = e^{-(\mu_y + \lambda)\cdot t}$$
$${}_t p_{xy} = e^{-(\mu_x + \mu_y + \lambda)\cdot t}$$

Special Probability:
Pr((*x*) and (*y*) die within *n* years by the common shock)

$$= \int_0^n {}_t p_{xy} \cdot \lambda dt = \int_0^n \lambda \cdot e^{-(\mu_x + \mu_y + \lambda)\cdot t} dt = \frac{\lambda}{\mu_x + \mu_y + \lambda} \cdot \left(1 - e^{-(\mu_x + \mu_y + \lambda)\cdot n}\right)$$

Insurance Present Value Random Variables
Single Life

Insurance Payable At The End Of The Year Of Death

(unless otherwise stated, benefit amount is 1 and age at issue is *x*)
(PVRV = Present Value Random Variable)
(SBP = Single Benefit Premium)
(APV = Actuarial Present Value)

1. whole life insurance

The probability distribution table for the PVRV, Z_x, is

Z_x	Probability
v	Pr(*K* = 0) = q_x
v^2	Pr(*K* = 1) = ${}_{1|}q_x$
v^3	Pr(*K* = 2) = ${}_{2|}q_x$
$\vdots$	$\vdots$

PVRV = $Z_x = v^{K+1}$

SBP = APV = $E[Z_x] = E[v^{K+1}] = A_x = vq_x + v^2\,{}_{1|}q_x + \cdots$

$E[Z_x^2] = {}^2A_x = v^2 q_x + v^4\,{}_{1|}q_x + \cdots$. Therefore, $Var(Z_x) = Var(v^{K+1}) = {}^2A_x - (A_x)^2$

Comments and Concepts: (Applies to all insurances in this section.)

(*i*) ${}^2A$ means to perform the same calculation as with A, except use double the force of interest. We will generally have for insurance that if *Z* is the PVRV, then *E*[*Z*] = *A* and $E[Z^2] = {}^2A$. This will **not** be the situation for annuities.

(*ii*) We can calculate probabilities involving the random variable *Z* by rewriting the event in terms of the random variable *K*.

2. *n*-year term insurance (benefit is paid if death occurs within next *n* years)

The probability distribution table for the PVRV, $Z^{1}_{x:\overline{n}|}$, is

$Z^{1}_{x:\overline{n}|}$	Probability
v	$\Pr(K = 0) = q_x$
v^2	$\Pr(K = 1) = {}_{1|}q_x$
v^3	$\Pr(K = 2) = {}_{2|}q_x$
$\vdots$	$\vdots$
v^n	$\Pr(K = n-1) = {}_{n-1|}q_x$
0	$\Pr(K \geq n) = {}_{n}p_x$

PVRV = $Z^{1}_{x:\overline{n}|} = \begin{cases} v^{K+1} \cdots K < n \\ 0 \cdots K \geq n \end{cases}$

SBP = APV = $E[Z^{1}_{x:\overline{n}|}] = A^{1}_{x:\overline{n}|} = vq_x + v^2 {}_{1|}q_x + \cdots + v^n {}_{n-1|}q_x$

$E[(Z^{1}_{x:\overline{n}|})^2] = {}^2A^{1}_{x:\overline{n}|} = v^2 q_x + v^4 {}_{1|}q_x + \cdots + v^{2n} {}_{n-1|}q_x$.

Therefore, $Var(Z^{1}_{x:\overline{n}|}) = {}^2A^{1}_{x:\overline{n}|} - (A^{1}_{x:\overline{n}|})^2$

Comments and Concepts:

The symbol $\overset{1}{x}:\overline{n|}$ is based on the contingent probability notation from page 11 of these notes. Here, the “life” *y* is $\overline{n|}$, an *n*-year certain period. Observe that *n*-year term insurance pays a death benefit when the status $\overset{1}{x}:\overline{n|}$ fails. That is, the death benefit is paid on the death of (*x*) as long as this death occurs before the death of the *n*-year certain period (within *n* years).

3. *n*-year pure endowment (benefit is paid if participant survives *n* years)

The probability distribution table for the PVRV, $Z_{x:\overset{1}{\overline{n|}}}$, is

$Z_{x:\overset{1}{\overline{n|}}}$	Probability
0	$\Pr(K < n) = {}_n q_x$
v^n	$\Pr(K \geq n) = {}_n p_x$

PVRV = $Z_{x:\overset{1}{\overline{n|}}} = \begin{cases} 0 \cdots K < n \\ v^n \cdots K \geq n \end{cases}$

SBP = APV = $E[Z_{x:\overset{1}{\overline{n|}}}] = A_{x:\overset{1}{\overline{n|}}} = v^n {}_n p_x$ (Another notation: ${}_n E_x = A_{x:\overset{1}{\overline{n|}}} = v^n {}_n p_x$)

$E[(Z_{x:\overset{1}{\overline{n|}}})^2] = {}^2A_{x:\overset{1}{\overline{n|}}} = v^{2n} {}_n p_x$. Therefore, $Var(Z_{x:\overset{1}{\overline{n|}}}) = {}^2A_{x:\overset{1}{\overline{n|}}} - (A_{x:\overset{1}{\overline{n|}}})^2$

Comments and Concepts:

The symbol $x:\overset{1}{\overline{n|}}$ is based on the contingent probability notation from page 11 of these notes. Observe that an *n*-year pure endowment pays when the status $x:\overset{1}{\overline{n|}}$ fails. That is, the benefit is paid on the death of the *n*-year certain period, $\overline{n|}$, as long as this death occurs before the death of (*x*). This is equivalent to saying the benefit is paid after *n* years, as long as (*x*) survives that long.

A very often tested recursion formula is:

$$A_x = A^{1}_{x:\overline{n|}} + {}_nE_x \cdot A_{x+n}$$

4. n-year endowment insurance (benefit is paid at end of year of death if participant dies before age $x + n$, and benefit is paid at age $x + n$ if participant survives to age $x + n$)

The probability distribution table for the PVRV, $Z_{x:\overline{n}|}$, is

$Z_{x:\overline{n}|}$	Probability
v	$\Pr(K = 0) = q_x$
v^2	$\Pr(K = 1) = {}_{1|}q_x$
v^3	$\Pr(K = 2) = {}_{2|}q_x$
$\vdots$	$\vdots$
v^n	$\Pr(K = n-1) = {}_{n-1|}q_x$
v^n	$\Pr(K \geq n) = {}_n p_x$

$$\text{PVRV} = Z_{x:\overline{n}|} = \begin{cases} v^{K+1} \cdots K < n \\ v^n \cdots K \geq n \end{cases} = Z^{1}_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\ \ 1}$$

$$\text{SBP} = \text{APV} = E[Z_{x:\overline{n}|}] = A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$$

$E[(Z_{x:\overline{n}|})^2] = {}^2A_{x:\overline{n}|} = {}^2A^{1}_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1}$. Therefore, $Var(Z_{x:\overline{n}|}) = {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2$

Comments and Concepts:

(i) Observe that n-year endowment insurance pays when the joint life status $x:\overline{n}|$ fails. The benefit is guaranteed to be paid and will be paid at the earlier of the death of (x) and the death of $\overline{n}|$.

(ii) On a random variable level, $(Z_{x:\overline{n}|})^2 = (Z^{1}_{x:\overline{n}|})^2 + (Z_{x:\overline{n}|}^{\ \ 1})^2$ since $Z^{1}_{x:\overline{n}|} \cdot Z_{x:\overline{n}|}^{\ \ 1} = 0$. So the expectation calculations above should be clear.

(iii) WARNING: $Var(Z_{x:\overline{n}|}) \neq Var(Z^{1}_{x:\overline{n}|}) + Var(Z_{x:\overline{n}|}^{\ \ 1})$ It is easy to show that if X and Y are random variables such that their product is 0, then $Cov(X,Y) = -\mu_X \cdot \mu_Y$. Therefore the correct variance formula is

$$Var(Z_{x:\overline{n}|}) = Var(Z^{1}_{x:\overline{n}|}) + Var(Z_{x:\overline{n}|}^{\ \ 1}) - 2\,A^{1}_{x:\overline{n}|}\,A_{x:\overline{n}|}^{\ \ 1}$$

(iv) With $n = 1$, we get $A_{x:\overline{1}|} = v \cdot q_x + v \cdot p_x = v(q_x + p_x) = v$

5. n-year deferred whole life insurance (benefit is paid at end of year of death if participant dies after age $x + n$, no benefit is paid if participant dies prior to age $x + n$)

The probability distribution table for the PVRV, ${}_{n|}Z_x$, is

${}_{n\|}Z_x$	Probability
0	$\Pr(K < n) = {}_n q_x$
v^{n+1}	$\Pr(K = n) = {}_{n\|}q_x$
v^{n+2}	$\Pr(K = n + 1) = {}_{n+1\|}q_x$
$\vdots$	$\vdots$

PVRV = ${}_{n|}Z_x = \begin{cases} 0 \cdots K < n \\ v^{K+1} \cdots K \geq n \end{cases} = Z_x - Z^{1}_{x:\overline{n|}}$

SBP = APV = $E[{}_{n|}Z_x] = {}_{n|}A_x = A_x - A^{1}_{x:\overline{n|}}$

$E[({}_{n|}Z_x)^2] = {}^2_{n|}A_x = {}^2A_x - {}^2A^{1}_{x:\overline{n|}}$. Therefore, $Var({}_{n|}Z_x) = {}^2_{n|}A_x - ({}_{n|}A_x)^2$

Comments and Concepts:

(i) We also have the following APV formula: ${}_{n|}A_x = v^n \, {}_n p_x A_{x+n}$

(ii) Remembering the meaning of ${}^2A$, we also get ${}^2_{n|}A_x = v^{2n} \cdot {}_n p_x \cdot {}^2A_{x+n}$

(iii) The ideas above can be extended to an n-year deferred, j-year term insurance, or an n-year deferred, j-year pure endowment, or a … If you are tested on one of these other insurance types, just use the basic principals learned by studying the above insurances.

6. whole life increasing insurance beginning at 1

The probability distribution table for the PVRV, $(IZ)_x$, is

$(IZ)_x$	Probability
v	$\Pr(K=0) = q_x$
$2v^2$	$\Pr(K=1) = {}_{1\|}q_x$
$3v^3$	$\Pr(K=2) = {}_{2\|}q_x$
$\vdots$	$\vdots$

PVRV = $(IZ)_x = (K+1)\cdot v^{K+1}$

SBP = APV = $E[(IZ)_x] = (IA)_x = A_x + {}_{1|}A_x + {}_{2|}A_x + \cdots$

7. n-year term increasing insurance beginning at 1

The probability distribution table for the PVRV, $(IZ)^{1}_{x:\overline{n}|}$, is

$(IZ)^{1}_{x:\overline{n}\|}$	Probability
v	$\Pr(K=0) = q_x$
$2v^2$	$\Pr(K=1) = {}_{1\|}q_x$
$3v^3$	$\Pr(K=2) = {}_{2\|}q_x$
$\vdots$	$\vdots$
nv^n	$\Pr(K = n-1) = {}_{n-1\|}q_x$
0	$\Pr(K \ge n) = {}_n p_x$

PVRV = $(IZ)^{1}_{x:\overline{n}|} = \begin{cases} (K+1)\cdot v^{K+1} \cdots K < n \\ 0 \cdots K \ge n \end{cases}$

SBP = APV =

$$E[(IZ)^{1}_{x:\overline{n}|}] = (IA)^{1}_{x:\overline{n}|} = v\cdot q_x + 2v^2\cdot p_x\cdot q_{x+1} + 3v^3\cdot {}_2p_x\cdot q_{x+1} + \cdots + nv^n\cdot {}_{n-1}p_x\cdot q_{n+n-1}$$

8. *n*-year decreasing insurance to 1

The probability distribution table for the PVRV, $(DZ)^{1}_{x:\overline{n}|}$, is

$(DZ)^{1}_{x:\overline{n}|}$	Probability
nv	Pr(K = 0) = q_x
$(n-1)v^2$	Pr(K = 1) = $_{1|}q_x$
$(n-2)v^3$	Pr(K = 2) = $_{2|}q_x$
$\vdots$	$\vdots$
v^n	Pr(K = $n-1$) = $_{n-1|}q_x$
0	Pr($K \geq n$) = $_n p_x$

PVRV = $(DZ)^{1}_{x:\overline{n}|} = \begin{cases} (n-K)\cdot v^{K+1} \cdots K < n \\ 0 \cdots K \geq n \end{cases}$

SBP = APV = $E[(DZ)^{1}_{x:\overline{n}|}] = (DA)^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A^{1}_{x:\overline{n-1}|} + A^{1}_{x:\overline{n-2}|} + \cdots + A^{1}_{x:\overline{1}|}$

Remark: $(DA)^{1}_{x:\overline{n}|} + (IA)^{1}_{x:\overline{n}|} = (n+1)\, A^{1}_{x:\overline{n}|}$

Insurance Payable At The Moment Of Death

1. whole life insurance

   PVRV = $\bar{Z}_x = v^T = e^{-\delta \cdot T}$

   SBP = APV = $E[\bar{Z}_x] = \bar{A}_x = \int_0^\infty e^{-\delta \cdot t} \cdot {}_t p_x \cdot \mu_x(t) dt$

   $E[\bar{Z}_x^2] = {}^2\bar{A}_x = \int_0^\infty e^{-2\delta \cdot t} \cdot {}_t p_x \cdot \mu_x(t) dt$

   Therefore, $Var(\bar{Z}_x) = {}^2\bar{A}_x - (\bar{A}_x)^2$.

   Important Comment on Notation: We are using a bar over *Z* to denote a different random variable than the random variable *Z*. Sometimes a bar over a random variable indicates the mean of a random sample from the same distribution as the random variable. That's not the case here. What we mean by using the bar notation is that the insurance is paid at the time of death, and thus the PVRV is a function of the complete future lifetime random variable, *T*.

2. *n*-year term insurance

   PVRV = $\bar{Z}^{1}_{x:\overline{n|}} = \begin{cases} v^T \cdots T < n \\ 0 \cdots T \geq n \end{cases}$

   SBP = APV = $E[\bar{Z}^{1}_{x:\overline{n|}}] = \bar{A}^{1}_{x:\overline{n|}} = \int_0^n v^t \cdot f_T(t) dt = \int_0^n e^{-\delta \cdot t} \cdot {}_t p_x \cdot \mu_x(t) dt$

   $E[(\bar{Z}^{1}_{x:\overline{n|}})^2] = {}^2\bar{A}^{1}_{x:\overline{n|}} = \int_0^n e^{-2\delta \cdot t} \cdot {}_t p_x \cdot \mu_x(t) dt$

   Therefore, $Var(\bar{Z}^{1}_{x:\overline{n|}}) = {}^2\bar{A}^{1}_{x:\overline{n|}} - (\bar{A}^{1}_{x:\overline{n|}})^2$.

3. *n*-year endowment insurance

PVRV = $\overline{Z}_{x:\overline{n}|} = \begin{cases} v^T \cdots T < n \\ v^n \cdots T \geq n \end{cases} = \overline{Z}^{1}_{x:\overline{n}|} + Z_{x:\overset{1}{\overline{n}|}}$

SBP = APV = $E[\overline{Z}_{x:\overline{n}|}] = \overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$

$E[(\overline{Z}_{x:\overline{n}|})^2] = {}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^{1}_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}}$

Therefore, $Var(\overline{Z}_{x:\overline{n}|}) = {}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2$.

Comments and Concepts: (Comments (*ii*) – (*iv*) are analogues to the same numbered comments in the *n*-year endowment insurance payable at the end of the year of death case – "the discrete case".)

(*i*) Notice that the pure endowment random variable is always discrete, and so we do not have a $\overline{Z}_{x:\overset{1}{\overline{n}|}}$ random variable. Subsequently, there are no actuarial symbols $\overline{A}_{x:\overset{1}{\overline{n}|}}$ or ${}^2\overline{A}_{x:\overset{1}{\overline{n}|}}$.

(*ii*) On a random variable level, $(\overline{Z}_{x:\overline{n}|})^2 = (\overline{Z}^{1}_{x:\overline{n}|})^2 + (Z_{x:\overset{1}{\overline{n}|}})^2$ since $\overline{Z}^{1}_{x:\overline{n}|} \cdot Z_{x:\overset{1}{\overline{n}|}} = 0$. So the expectation calculations above should be clear.

(*iii*) WARNING: $Var(\overline{Z}_{x:\overline{n}|}) \neq Var(\overline{Z}^{1}_{x:\overline{n}|}) + Var(Z_{x:\overset{1}{\overline{n}|}})$ It is easy to show that if *X* and *Y* are random variables such that their product is 0, then $Cov(X,Y) = -\mu_X \cdot \mu_Y$. Therefore the correct variance formula is

$Var(\overline{Z}_{x:\overline{n}|}) = Var(\overline{Z}^{1}_{x:\overline{n}|}) + Var(Z_{x:\overset{1}{\overline{n}|}}) - 2\overline{A}^{1}_{x:\overline{n}|}\, A_{x:\overset{1}{\overline{n}|}}$

(*iv*) We do not get the nice formula in the $n = 1$ case that we got in the discrete case.

4. *n*-year deferred whole life insurance

PVRV = ${}_{n|}\bar{Z}_x = \begin{cases} 0 \cdots T < n \\ v^T \cdots T \geq n \end{cases} = \bar{Z}_x - \bar{Z}^{\,1}_{x:\overline{n|}}$

SBP = APV = $E[{}_{n|}\bar{Z}_x] = {}_{n|}\bar{A}_x = \bar{A}_x - \bar{A}^{\,1}_{x:\overline{n|}}$

$E[({}_{n|}\bar{Z}_x)^2] = {}^2_{n|}\bar{A}_x = {}^2\bar{A}_x - {}^2\bar{A}^{\,1}_{x:\overline{n|}}$

Therefore, $Var({}_{n|}\bar{Z}_x) = {}^2_{n|}\bar{A}_x - ({}_{n|}\bar{A}_x)^2$.

Comments and Concepts: (These comments are analogues to the same numbered comments in the *n*-year deferred whole life insurance payable at the end of the year of death case – "the discrete case".)

(*i*) We also have the following APV formula: ${}_{n|}\bar{A}_x = v^n \, {}_n p_x \, \bar{A}_{x+n}$

(*ii*) Remembering the meaning of ${}^2A$, we also get ${}^2_{n|}\bar{A}_x = v^{2n} \cdot {}_n p_x \cdot {}^2\bar{A}_{x+n}$

(*iii*) The ideas above can be extended to an *n*-year deferred, *j*-year term insurance, or an *n*-year deferred, *j*-year pure endowment, or a … If you are tested on one of these other insurance types, just use the basic principals learned by studying the above insurances.

Often Tested 1-Year Recursion Relationships

$$A_x = A^{1}_{x:\overline{1|}} + vp_x A_{x+1} = vq_x + vp_x A_{x+1}$$

$$A^{1}_{x:\overline{n|}} = A^{1}_{x:\overline{1|}} + vp_x\, A^{1}_{x+1:\overline{n-1|}} = vq_x + vp_x\, A^{1}_{x+1:\overline{n-1|}}$$

$$A_{x:\overline{n|}} = A^{1}_{x:\overline{1|}} + vp_x\, A_{x+1:\overline{n-1|}} = vq_x + vp_x\, A_{x+1:\overline{n-1|}}$$

The next three are the continuous analogues to the previous three.

$$\overline{A}_x = \overline{A}^{1}_{x:\overline{1|}} + vp_x \overline{A}_{x+1}$$

$$\overline{A}^{1}_{x:\overline{n|}} = \overline{A}^{1}_{x:\overline{1|}} + vp_x \overline{A}^{1}_{x+1:\overline{n-1|}}$$

$$\overline{A}_{x:\overline{n|}} = \overline{A}^{1}_{x:\overline{1|}} + vp_x \overline{A}_{x+1:\overline{n-1|}}$$

Non-unit Insurance (benefit payable at time *T* is b_T)

PVRV = $Z = b_T v^T$ (whole life)

$$Z = \begin{cases} b_T v^T \cdots T < n \\ 0 \cdots T \geq n \end{cases} \quad (n\text{-year term})$$

$\vdots$

$\vdots$

$$Z^2 = b_T^2 v^{2T} \quad \text{(whole life)}$$

$$Z^2 = \begin{cases} b_T^2 v^{2T} \cdots T < n \\ 0 \cdots T \geq n \end{cases} \quad (n\text{-year term})$$

$\vdots$

$\vdots$

Notice that $E[Z^2]$ will not equal ${}^2A$ unless the benefit is 1.

Annuity Present Value Random Variables
Single Life

Life Annuities (Due) Payable At The Beginning Of The Year

(all the following annuities have an annual payment of 1)

1. whole life annuity due

The probability distribution table for the PVRV, $\ddot{Y}_x$, is

$\ddot{Y}_x$	Probability
$\ddot{a}_{\overline{1}\rceil}$	$\Pr(K = 0) = q_x$
$\ddot{a}_{\overline{2}\rceil}$	$\Pr(K = 1) = {}_{1\vert}q_x$
$\vdots$	$\vdots$

$$\text{PVRV} = \ddot{Y}_x = \ddot{a}_{\overline{K+1}\rceil} = \frac{1-v^{K+1}}{d} = \frac{1-Z_x}{d}$$

$$\text{SBP} = \text{APV} = E[\ddot{Y}_x] = \ddot{a}_x = \ddot{a}_{\overline{1}\rceil} \cdot q_x + \ddot{a}_{\overline{2}\rceil} \cdot {}_{1\vert}q_x + \cdots = 1 + vp_x + v^2 \cdot {}_2p_x + \cdots$$

Comments and Concepts:

(i) $\ddot{a}_x = E[\ddot{Y}_x] = E\left[\frac{1-Z_x}{d}\right] = \frac{1-A_x}{d}$ (equivalently, $A_x = 1 - d \cdot \ddot{a}_x$)

(ii) $Var(\ddot{Y}_x) = \frac{1}{d^2} \cdot Var(Z_x) = \frac{1}{d^2} \cdot \left({}^2A_x - (A_x)^2\right)$

(iii) We can calculate probabilities involving the random variable Y by rewriting the event in terms of the random variable K.

2. *n*-year temporary life annuity due (pays 1 until the earlier of the death of the annuitant or an *n*-year certain period)

   The probability distribution table for the PVRV, $\ddot{Y}_{x:\overline{n}|}$, is

$\ddot{Y}_{x:\overline{n}|}$	Probability
$\ddot{a}_{\overline{1}|}$	$\Pr(K = 0) = q_x$
$\ddot{a}_{\overline{2}|}$	$\Pr(K = 1) = {}_{1|}q_x$
$\ddot{a}_{\overline{3}|}$	$\Pr(K = 2) = {}_{2|}q_x$
$\vdots$	$\vdots$
$\ddot{a}_{\overline{n}|}$	$\Pr(K = n-1) = {}_{n-1|}q_x$
$\ddot{a}_{\overline{n}|}$	$\Pr(K \geq n) = {}_n p_x$

$$\text{PVRV} = \ddot{Y}_{x:\overline{n}|} = \begin{cases} \ddot{a}_{\overline{K+1}|} \cdots K < n \\ \ddot{a}_{\overline{n}|} \cdots K \geq n \end{cases} = \begin{cases} \dfrac{1-v^{K+1}}{d} \cdots K < n \\ \dfrac{1-v^{n}}{d} \cdots K \geq n \end{cases} = \frac{1-Z_{x:\overline{n}|}}{d}$$

SBP = APV =

$$E[\ddot{Y}_{x:\overline{n}|}] = \ddot{a}_{x:\overline{n}|} = \ddot{a}_{\overline{1}|} \cdot q_x + \cdots + \ddot{a}_{\overline{n}|} \cdot {}_{n-1|}q_x + \ddot{a}_{\overline{n}|} \cdot {}_n p_x = 1 + vp_x + v^2 \cdot {}_2 p_x + \cdots + v^{n-1} \cdot {}_{n-1} p_x$$

Comments and Concepts:

(i) $\ddot{a}_{x:\overline{n}|} = E[\ddot{Y}_{x:\overline{n}|}] = E\left[\dfrac{1-Z_{x:\overline{n}|}}{d}\right] = \dfrac{1-A_{x:\overline{n}|}}{d}$ (equivalently, $A_{x:\overline{n}|} = 1 - d \cdot \ddot{a}_{x:\overline{n}|}$)

(ii) $Var(\ddot{Y}_{x:\overline{n}|}) = \dfrac{1}{d^2} \cdot Var(Z_{x:\overline{n}|}) = \dfrac{1}{d^2} \cdot \left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$

(iii) We denote the actuarial accumulated value (AAV) of an *n*-year temporary annuity due by $\ddot{s}_{x:\overline{n}|}$ and define it by the relationship

$\ddot{a}_{x:\overline{n}|} = \ddot{s}_{x:\overline{n}|} \cdot v^n \cdot {}_n p_x$. That is, $\ddot{s}_{x:\overline{n}|} = \dfrac{\ddot{a}_{x:\overline{n}|}}{v^n \cdot {}_n p_x} = \dfrac{\ddot{a}_{x:\overline{n}|}}{{}_n E_x} = \dfrac{\ddot{a}_{x:\overline{n}|}}{A_{x:\overset{1}{\overline{n}|}}}$.

(A more complete discussion of AAV is found later in these notes.)

3. *n*-year deferred life annuity due

The probability distribution table for the PVRV, ${}_{n|}\ddot{Y}_x$, is

${}_{n\|}\ddot{Y}_x$	Probability
0	$\Pr(K < n) = {}_n q_x$
${}_{n\|}\ddot{a}_{\overline{1}\|}$	$\Pr(K = n) = {}_{n\|} q_x$
${}_{n\|}\ddot{a}_{\overline{2}\|}$	$\Pr(K = n+1) = {}_{n+1\|} q_x$
$\vdots$	$\vdots$

$$\text{PVRV} = {}_{n|}\ddot{Y}_x = {}_{n|}\ddot{a}_{\overline{K+1-n}|} = \begin{cases} 0 \cdots K < n \\ \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|} \cdots K \ge n \end{cases} = \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|}$$

$$\text{SBP} = \text{APV} = E[{}_{n|}\ddot{Y}_x] = {}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n \cdot {}_n p_x + v^{n+1} \cdot {}_{n+1} p_x + \cdots$$

Comments and Concepts:

(i) ${}_{n|}\ddot{a}_x = v^n \cdot {}_n p_x \cdot \ddot{a}_{x+n}$

(ii) $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + v^n \cdot {}_n p_x \cdot \ddot{a}_{x+n}$

(iii) $E[({}_{n|}\ddot{Y}_x)^2] = \frac{2}{d} \cdot v^{2n} \cdot {}_n p_x \cdot \left(\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n}\right) + {}^2_{n|}\ddot{a}_x$ (unlikely to see this on the exam)

4. *n*-year certain and life annuity due (pays 1 until the later of the death of the annuitant or an *n*-year certain period)

   The probability distribution table for the PVRV, $\ddot{Y}_{\overline{\overline{x:\overline{n}|}}}$, is

$\ddot{Y}_{\overline{\overline{x:\overline{n}|}}}$	Probability
$\ddot{a}_{\overline{n}|}$	$\Pr(K < n) = {}_n q_x$
$\ddot{a}_{\overline{n+1}|}$	$\Pr(K = n) = {}_{n|} q_x$
$\ddot{a}_{\overline{n+2}|}$	$\Pr(K = n+1) = {}_{n+1|} q_x$
$\vdots$	$\vdots$

   $$\text{PVRV} = \ddot{Y}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{Y}_x = \begin{cases} \ddot{a}_{\overline{n}|} \cdots K < n \\ \ddot{a}_{\overline{K+1}|} \cdots K \geq n \end{cases}$$

   $$\text{SBP} = \text{APV} = E[\ddot{Y}_{\overline{\overline{x:\overline{n}|}}}] = \ddot{a}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$$

   Comments and Concepts:

   Observe that an *n*-year certain and life annuity due pays until the failure of the last-survivor status $\overline{x:\overline{n}|}$ (here, the "life" *y* is $\overline{n}|$, an *n*-year certain period). That is, the annuity pays until the later of the death of (*x*) and the death of $\overline{n}|$, or equivalently, it pays as long as (*x*) is alive, with a minimum of *n* years.

Life Annuities (Immediate) Payable At The End Of The Year
(all the following annuities have an annual payment of 1)

Comments and Concepts:

When dealing with annuities immediate, use the above formulas for annuities due and relationships between annuities immediate and annuities due.

Examples:

1. If Y_x is the PVRV for a whole life annuity immediate, then $Y_x = \ddot{Y}_x - 1$.
   Then SBP = APV = $a_x = E[Y_x] = E[\ddot{Y}_x - 1] = \ddot{a}_x - 1 = v \cdot p_x + v^2 \cdot {}_2p_x + \cdots$ and $Var(Y_x) = Var(\ddot{Y}_x)$.

   Important Special Formula:
   If i = 0, then $a_x = e_x = E[K(x)]$

2. If $Y_{x:\overline{n}|}$ is the PVRV for an n-year temporary life annuity immediate, then $Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n+1}|} - 1$. Then SBP = APV =
   $a_{x:\overline{n}|} = E[Y_{x:\overline{n}|}] = E[\ddot{Y}_{x:\overline{n+1}|} - 1] = \ddot{a}_{x:\overline{n+1}|} - 1 = v \cdot p_x + v^2 \cdot {}_2p_x + \cdots v^n \cdot {}_np_x$ and
   $Var(Y_{x:\overline{n}|}) = Var(\ddot{Y}_{x:\overline{n+1}|})$

   Important Special Formulas:
   (i) If i = 0, then $a_{x:\overline{n}|} = e_{x:\overline{n}|}$
   (ii) $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \cdot {}_np_x = \ddot{a}_{x:\overline{n}|} - 1 + {}_nE_x$

3. If ${}_{n|}Y_x$ is the PVRV for an n-year deferred life annuity immediate, then ${}_{n|}Y_x = {}_{n+1|}\ddot{Y}_x$. Then SBP = APV = $E[{}_{n|}Y_x] = {}_{n|}a_x = {}_{n+1|}\ddot{a}_x$ and
   $Var({}_{n|}Y_x) = Var({}_{n+1|}\ddot{Y}_x)$

Continuous Life Annuities
(all the following annuities have an annual payment rate of 1)

1. whole life continuous annuity

PVRV = $\overline{Y}_x = \overline{a}_{\overline{T|}} = \dfrac{1-v^T}{\delta} = \dfrac{1-\overline{Z}_x}{\delta}$

SBP = APV = $E[\overline{Y}_x] = \overline{a}_x = \int_0^\infty \overline{a}_{\overline{t|}} \cdot {}_t p_x \cdot \mu_x(t)dt = \int_0^\infty v^t \cdot {}_t p_x dt$

Comments and Concepts:

(i) $\overline{a}_x = E[\overline{Y}_x] = E\left[\dfrac{1-\overline{Z}_x}{\delta}\right] = \dfrac{1-\overline{A}_x}{\delta}$ (equivalently, $\overline{A}_x = 1-\delta\cdot\overline{a}_x$)

(ii) $Var(\overline{Y}_x) = \dfrac{1}{\delta^2}\cdot Var(\overline{Z}_x) = \dfrac{1}{\delta^2}\cdot\left({}^2\overline{A}_x - (\overline{A}_x)^2\right)$

(iii) If i = 0, then $\overline{a}_x = \overset{0}{e}_x$.

Important Comment on Notation: Just as with the case of insurance payable at the moment of death, we are using a bar over *Y* to denote a different random variable than the random variable *Y*. What we mean by using the bar notation is that the annuity is paid continuously, and thus the PVRV is a function of the complete future lifetime random variable, *T*.

2. *n*-year temporary life continuous annuity

PVRV = $\overline{Y}_{x:\overline{n}|} = \begin{cases} \overline{a}_{\overline{T}|} \cdots T < n \\ \overline{a}_{\overline{n}|} \cdots T \geq n \end{cases} = \dfrac{1-\overline{Z}_{x:\overline{n}|}}{\delta}$

SBP = APV = $E[\overline{Y}_{x:\overline{n}|}] = \overline{a}_{x:\overline{n}|} = \int_0^n \overline{a}_{\overline{t}|} \cdot {}_t p_x \cdot \mu_x(t)dt + \overline{a}_{\overline{n}|} \cdot {}_n p_x = \int_0^n v^t \cdot {}_t p_x dt$

Comments and Concepts:

(i) $\overline{a}_{x:\overline{n}|} = E[\overline{Y}_{x:\overline{n}|}] = E\left[\dfrac{1-\overline{Z}_{x:\overline{n}|}}{\delta}\right] = \dfrac{1-\overline{A}_{x:\overline{n}|}}{\delta}$ (equivalently, $\overline{A}_{x:\overline{n}|} = 1 - \delta \cdot \overline{a}_{x:\overline{n}|}$)

(ii) $Var(\overline{Y}_{x:\overline{n}|}) = \dfrac{1}{\delta^2} \cdot Var(\overline{Z}_{x:\overline{n}|}) = \dfrac{1}{\delta^2} \cdot \left({}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2\right)$

(iii) If $i = 0$, then $\overline{a}_{x:\overline{n}|} = \overset{0}{e}_{x:\overline{n}|}$.

(iv) We denote the actuarial accumulated value (AAV) of an *n*-year temporary continuous annuity by $\overline{s}_{x:\overline{n}|}$ and define it by the relationship $\overline{a}_{x:\overline{n}|} = \overline{s}_{x:\overline{n}|} \cdot v^n \cdot {}_n p_x$. Then $\overline{s}_{x:\overline{n}|} = \dfrac{\overline{a}_{x:\overline{n}|}}{v^n \cdot {}_n p_x} = \dfrac{\overline{a}_{x:\overline{n}|}}{{}_n E_x} = \dfrac{\overline{a}_{x:\overline{n}|}}{A_{x:\overset{1}{\overline{n}|}}}$.

(A more complete discussion of AAV is found later in these notes.)

3. *n*-year deferred whole life continuous annuity

PVRV = $_{n|}\overline{Y}_x = \begin{cases} 0 \cdots T < n \\ \overline{a}_{\overline{T}|} - \overline{a}_{\overline{n}|} \cdots T \geq n \end{cases} = \begin{cases} 0 \cdots T < n \\ v^n \cdot \overline{a}_{\overline{T-n}|} \cdots T \geq n \end{cases} = \overline{Y}_x - \overline{Y}_{x:\overline{n}|} = \dfrac{\overline{Z}_{x:\overline{n}|} - \overline{Z}_x}{\delta}$

SBP = APV = $E[{}_{n|}\overline{Y}_x] = {}_{n|}\overline{a}_x = \overline{a}_x - \overline{a}_{x:\overline{n}|} = \int_n^\infty v^n \cdot \overline{a}_{\overline{t-n}|} \cdot {}_t p_x \cdot \mu_x(t) dt = \int_n^\infty v^t \cdot {}_t p_x dt$

Comments and Concepts:

(i) $\quad {}_{n|}\overline{a}_x = v^n \cdot {}_n p_x \cdot \overline{a}_{x+n}$

(ii) $\quad \overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n|}\overline{a}_x = \overline{a}_{x:\overline{n}|} + v^n \cdot {}_n p_x \cdot \overline{a}_{x+n}$

(iii) $\quad E[({}_{n|}\overline{Y}_x)^2] = \dfrac{2}{\delta} \cdot v^{2n} \cdot {}_n p_x \cdot \left(\overline{a}_{x+n} - {}^2\overline{a}_{x+n} \right)$ (unlikely to see this on the exam)

4. *n*-year certain and life continuous annuity

PVRV = $\overline{Y}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n}|} + {}_{n|}\overline{Y}_x = \begin{cases} \overline{a}_{\overline{n}|} \cdots T < n \\ \overline{a}_{\overline{T}|} \cdots T \geq n \end{cases}$

SBP = APV = $E[\overline{Y}_{\overline{x:\overline{n}|}}] = \overline{a}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n}|} + {}_{n|}\overline{a}_x$

Life Annuities Due Payable mthly

These annuities also have an annual payment of 1. Thus the symbol $C \cdot \ddot{a}_x^{(m)}$ represents the APV of a life annuity to (*x*) that pays $\frac{C}{m}$ at the beginning of each of *m* periods per year for the life of (*x*). The total annual payment each year is *C*.

Under the UDD assumption we have:

$$\ddot{a}_x^{(m)} = \alpha(m) \cdot \ddot{a}_x - \beta(m) \quad \text{and} \quad \ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m) \cdot \ddot{a}_{x:\overline{n}|} - \beta(m) \cdot \left(1 - {}_nE_x\right)$$

where the values of $\alpha(m)$ and $\beta(m)$ will be given in the tables

Often Tested 1-Year Recursion Relationships

$$\ddot{a}_x = 1 + v \cdot p_x \cdot \ddot{a}_{x+1}$$
$$\ddot{a}_{x:\overline{n}|} = 1 + v \cdot p_x \cdot \ddot{a}_{x+1:\overline{n-1}|}$$
$$\bar{a}_x = \bar{a}_{x:\overline{1}|} + v \cdot p_x \cdot \bar{a}_{x+1}$$
$$\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{1}|} + v \cdot p_x \cdot \bar{a}_{x+1:\overline{n-1}|}$$
$$a_x = v \cdot p_x + v \cdot p_x \cdot a_{x+1} = v \cdot p_x \cdot \left(1 + a_{x+1}\right)$$

Insurance and Annuity Present Value Random Variables
Multiple Life

Contingent Insurance Payable at the Moment of Death

Type 1: Pays 1 on the death of (*x*) if (*x*) dies first

$$\text{SBP} = \text{APV} = \bar{A}_{\overset{1}{x}y} = \int_0^\infty v^t \cdot {}_t p_{xy} \cdot \mu_x(t)dt$$

Type 2: Pays 1 on the death of (*x*) if (*x*) dies second

$$\text{SBP} = \text{APV} = \bar{A}_{\overset{2}{x}y} = \int_0^\infty v^t \cdot {}_t q_y \cdot {}_t p_x \cdot \mu_x(t)dt$$

Interchange the roles of (*x*) and (*y*) to get similar formulas for insurance payable on the death of (*y*).

Contingent insurance payable at the end of the year of death
(Use summations instead of integrals)

Contingent Insurance Relationships

1. $\bar{A}_x = \bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y}$

2. $\bar{A}_y = \bar{A}_{x\overset{1}{y}} + \bar{A}_{x\overset{2}{y}}$

3. $\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$

4. $\bar{A}_{\overline{xy}} = \bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}}$

There are similar relationships in the discrete case, i.e. no bars.

Continuous Insurance and Annuities for Joint-Life Status
(Replace *x* in all single life formulas by *xy*)

1. $\bar{A}_{xy} = E[v^{T(xy)}] = \int_0^\infty v^t \cdot {}_t p_{xy} \cdot \mu_{xy}(t)dt$

2. ${}^2\bar{A}_{xy} = E[v^{2T(xy)}] = \int_0^\infty v^{2t} \cdot {}_t p_{xy} \cdot \mu_{xy}(t)dt$

   So $Var(\bar{Z}_{xy}) = {}^2\bar{A}_{xy} - (\bar{A}_{xy})^2$ = variance of PVRV for insurance that pays 1 at the moment of the first death of (*x*) or (*y*)

3. $\bar{a}_{xy} = \int_0^\infty v^t \cdot {}_t p_{xy} dt = \dfrac{1-\bar{A}_{xy}}{\delta}$

4. $\bar{a}_{xy:\overline{n}|} = \int_0^n v^t \cdot {}_t p_{xy} dt = \dfrac{1-\bar{A}_{xy:\overline{n}|}}{\delta}$

   This is the APV of an *n*-year temporary life annuity that pays continuously at a rate of 1 per year for the joint lifetimes of (*x*) and (*y*). This annuity pays until failure of the $xy:\overline{n}|$ status, which fails at the first of the death of (*x*), the death of (*y*), and the death of $\overline{n}|$. That is, the annuity pays until the first of the death of (*x*) or the death of (*y*), up to a maximum of *n* years.

5. $\bar{A}^{\;1}_{xy:\overline{n}|} = \int_0^n v^t \cdot {}_t p_{xy} \cdot \mu_{xy}(t)dt$

6. $Var(\bar{Y}_{xy}) = \dfrac{1}{\delta^2}\left({}^2\bar{A}_{xy} - (\bar{A}_{xy})^2\right)$ = variance of PVRV for a continuous annuity that pays 1 per year for as long as the joint-life status (*xy*) survives, i.e. as long as both (*x*) and (*y*) are alive.

   ⋮

   You get the idea.

Discrete Insurance and Annuities for Joint-Life Status
(Replace *x* in all single life formulas by *xy*)

1. $A_{xy} = E[v^{K(xy)+1}] = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k|}q_{xy}$

2. ${}^2A_{xy} = E[v^{2(K(xy)+1)}] = \sum_{k=0}^{\infty} v^{2(k+1)} \cdot {}_{k|}q_{xy}$
   So $Var(Z_{xy}) = {}^2A_{xy} - (A_{xy})^2$ = variance of PVRV for insurance that pays 1 at the end of the year of the first of the death of (*x*) or the death of (*y*)

3. $A_{xy:\overline{n}|} = A^{1}_{xy:\overline{n}|} + v^n \cdot {}_n p_{xy} = \sum_{k=0}^{n-1} v^{k+1} \cdot {}_{k|}q_{xy} + v^n \cdot {}_n p_{xy}$

4. $\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{xy} = \dfrac{1 - A_{xy}}{d}$

5. $\ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_k p_{xy} = \dfrac{1 - A_{xy:\overline{n}|}}{d}$

6. $Var(\ddot{Y}_{xy:\overline{n}|}) = \dfrac{1}{d^2}\left({}^2A_{xy:\overline{n}|} - (A_{xy:\overline{n}|})^2\right)$ = variance of PVRV for an *n*-year temporary life annuity due that pays 1 at the beginning of each year that the joint-life status $xy:\overline{n}|$ survives, i.e. as long as both (*x*) and (*y*) are alive, up to a maximum of *n* years.
   ⋮
   You get the idea.

Insurance and Annuities for Last-Survivor Status

Method 1: Replace x in all single life formulas by $\overline{xy}$, or this is the same as replacing the subscript xy in the joint-life status formulas above by $\overline{xy}$.

Method 2: (This can make many computations much easier.) Make use of the following relationships.

Often Used Relationships Between Joint-Life and Last-Survivor Statuses

Note: $T(xy)+T(\overline{xy})=T(x)+T(y)$ and $T(xy)\cdot T(\overline{xy})=T(x)\cdot T(y)$

1. ${}_tp_{xy}+{}_tp_{\overline{xy}}={}_tp_x+{}_tp_y$

2. ${}_tq_{xy}+{}_tq_{\overline{xy}}={}_tq_x+{}_tq_y$

3. $\mathring{e}_{xy}+\mathring{e}_{\overline{xy}}=\mathring{e}_x+\mathring{e}_y$

4. $e_{xy}+e_{\overline{xy}}=e_x+e_y$

5. $A_{xy}+A_{\overline{xy}}=A_x+A_y$

6. $a_{xy}+a_{\overline{xy}}=a_x+a_y$

7. $A_{xy:\overline{n|}}+A_{\overline{xy}:\overline{n|}}=A_{x:\overline{n|}}+A_{y:\overline{n|}}$

8. $A^{1}_{xy:\overline{n|}}+A^{1}_{\overline{xy}:\overline{n|}}=A^{1}_{x:\overline{n|}}+A^{1}_{y:\overline{n|}}$

9. $a_{xy:\overline{n|}}+a_{\overline{xy}:\overline{n|}}=a_{x:\overline{n|}}+a_{y:\overline{n|}}$

10. ${}_{n|}a_{xy}+{}_{n|}a_{\overline{xy}}={}_{n|}a_x+{}_{n|}a_y$

11. $Cov(T(xy),T(\overline{xy}))=Cov(T(x),T(y))+\left(\mathring{e}_x-\mathring{e}_{xy}\right)\cdot\left(\mathring{e}_y-\mathring{e}_{xy}\right)$

There are formulas in the continuous case too ($\bar{a}$'s and $\bar{A}$'s), and there are other formulas, e.g. n-year deferred insurance, pure endowments, ….

Loss Random Variables and Reserves

We focus on the single life case with issue age x. The ideas can be extended to the multiple life case, as will be seen in some of the examples that we'll do.

Loss-at-issue Random Variable

Suppose insurance is purchased with annual premiums of Q. Then the random variable representing the loss-at-issue will be of the form $L = Z - Q \cdot Y$.

Examples:

1a. Fully Continuous Whole Life Insurance of 1 with premiums of Q for life

$$L = \overline{Z}_x - Q \cdot \overline{Y}_x = \overline{Z}_x - Q \cdot \frac{1-\overline{Z}_x}{\delta} = \left(1 + \frac{Q}{\delta}\right) \cdot \overline{Z}_x - \frac{Q}{\delta}$$

$$E[L] = \overline{A}_x - Q \cdot \overline{a}_x = \left(1 + \frac{Q}{\delta}\right) \cdot \overline{A}_x - \frac{Q}{\delta}$$

$$Var(L) = \left(1 + \frac{Q}{\delta}\right)^2 \cdot Var\left(\overline{Z}_x\right) = \left(1 + \frac{Q}{\delta}\right)^2 \cdot \left({}^2\overline{A}_x - (\overline{A}_x)^2\right)$$

1b. Fully Discrete Whole Life Insurance of 1 with premiums of Q for life

$$L = Z_x - Q \cdot Y_x = Z_x - Q \cdot \frac{1-Z_x}{d} = \left(1 + \frac{Q}{d}\right) \cdot Z_x - \frac{Q}{d}$$

$$E[L] = A_x - Q \cdot \ddot{a}_x = \left(1 + \frac{Q}{d}\right) \cdot A_x - \frac{Q}{d}$$

$$Var(L) = \left(1 + \frac{Q}{d}\right)^2 \cdot Var\left(Z_x\right) = \left(1 + \frac{Q}{d}\right)^2 \cdot \left({}^2A_x - (A_x)^2\right)$$

2a. Fully Continuous n-year Endowment Insurance of 1 with premiums of Q

$$L = \overline{Z}_{x:\overline{n|}} - Q \cdot \overline{Y}_{x:\overline{n|}} = \overline{Z}_{x:\overline{n|}} - Q \cdot \frac{1-\overline{Z}_{x:\overline{n|}}}{\delta} = \left(1 + \frac{Q}{\delta}\right) \cdot \overline{Z}_{x:\overline{n|}} - \frac{Q}{\delta}$$

$$E[L] = \overline{A}_{x:\overline{n|}} - Q \cdot \overline{a}_{x:\overline{n|}} = \left(1 + \frac{Q}{\delta}\right) \cdot \overline{A}_{x:\overline{n|}} - \frac{Q}{\delta}$$

$$Var(L) = \left(1 + \frac{Q}{\delta}\right)^2 \cdot Var\left(\overline{Z}_{x:\overline{n|}}\right) = \left(1 + \frac{Q}{\delta}\right)^2 \cdot \left({}^2\overline{A}_{x:\overline{n|}} - (\overline{A}_{x:\overline{n|}})^2\right)$$

2b. Fully Discrete *n*-year Endowment Insurance of 1 with premiums of *Q*

$$L = Z_{x:\overline{n}|} - Q \cdot \ddot{Y}_{x:\overline{n}|} = Z_{x:\overline{n}|} - Q \cdot \frac{1 - Z_{x:\overline{n}|}}{d} = \left(1 + \frac{Q}{d}\right) \cdot Z_{x:\overline{n}|} - \frac{Q}{d}$$

$$E[L] = A_{x:\overline{n}|} - Q \cdot \ddot{a}_{x:\overline{n}|} = \left(1 + \frac{Q}{d}\right) \cdot A_{x:\overline{n}|} - \frac{Q}{d}$$

$$Var(L) = \left(1 + \frac{Q}{d}\right)^2 \cdot Var\left(Z_{x:\overline{n}|}\right) = \left(1 + \frac{Q}{d}\right)^2 \cdot \left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

You get the idea. There are many combinations of the type of insurance purchased and the method of paying premiums. We use one more example to illustrate the concept of the loss-at-issue random variable.

3. Insurance – *n*-year term with benefit of 1 payable at the moment of death
   Premiums - paid at the beginning of each year for *h* ($\leq n$) years

   (This is called *h*-payment, *n*-year term insurance. During the first *h* years, premiums are paid while living and the benefit is paid upon death. During the next *n* – *h* years, the insurance is already paid in full and so no premiums are paid, but the benefit is still paid upon death. After *n* years, the policy has expired and no benefit will be paid upon death.)

$$L = \overline{Z}^{\,1}_{x:\overline{n}|} - Q \cdot \ddot{Y}_{x:\overline{h}|}$$

$$E[L] = \overline{A}^{\,1}_{x:\overline{n}|} - Q \cdot \ddot{a}_{x:\overline{h}|}$$

   (If you're asked about variance of this one, skip it. ☺)

Calculating Premiums

Method 1: Percentiles

The premium Q is found by solving a probability equation such as $\Pr(L > 0) = 0.05$ (probability of a positive loss is 5%). This equation can be solved by writing the event first in terms of the random variable Z and then in terms of the random variable T (or K). Then use the distribution of T (or K) to solve the probability equation. We will illustrate this with examples.

Method 2: Equivalence Principle

The premium when found using the equivalence principle is called the **benefit premium.**

The premium Q is found by solving the equation $E[L] = 0$. This is the same equation as the one obtained by setting the APV at issue of benefits equal to the APV at issue of premiums.

Examples and Notation:

1. For fully discrete insurance of 1, benefit premiums are

$$P_x = \frac{A_x}{\ddot{a}_x} = \text{whole life insurance with premiums paid for life}$$

$$P^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = n\text{-year term insurance with at most } n \text{ premiums}$$

$$_hP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}} = h\text{-payment } n\text{-year endowment insurance}$$

⋮

You get the idea.

Important Fact: $P_{x:\overline{n}|} = P^{1}_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ 1}$

2. For fully continuous insurance of 1, benefit premiums are

$\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_x}$ = whole life insurance with premiums paid for life

${}_k\bar{P}({}_{n|}\bar{A}_x) = \dfrac{{}_{n|}\bar{A}_x}{\bar{a}_{x:\overline{k|}}}$ = k-payment n-year deferred whole life insurance

⋮

You get the idea.

3. For semi-continuous insurance of 1, benefit premiums are

$P(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_x}$ = whole life insurance with premiums paid for life

${}_hP(\bar{A}^1_{x:\overline{n|}}) = \dfrac{\bar{A}^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{h|}}}$ = h-payment n-year term insurance

⋮

You get the idea. Notice the notation suggests, as is correct, that the benefit is paid at the moment of death, and the premiums are paid at the beginning of each year.

4. m[th]ly benefit premiums

$P_x^{(12)} = \dfrac{A_x}{\ddot{a}_x^{(12)}}$ = fully discrete whole life insurance with premiums of $\dfrac{P_x^{(12)}}{12}$ paid at the beginning of each month for life

$P^{(4)}(\bar{A}_{x:\overline{n|}}) = \dfrac{\bar{A}_{x:\overline{n|}}}{\ddot{a}^{(4)}_{x:\overline{n|}}}$ = semi-continuous n-year endowment insurance with premiums of $\dfrac{P^{(4)}(\bar{A}_{x:\overline{n|}})}{4}$ paid at the beginning of each quarter until death of the insurance expires

⋮

You get the idea.

Special Relationships When Using Benefit Premiums
Fully Discrete or Fully Continuous Whole Life or Endowment Insurance

For fully discrete or fully continuous whole life insurance or endowment insurance, the variance of the loss-at-issue random variable had a factor of $\left(1+\frac{Q}{d}\right)$ (discrete case) or $\left(1+\frac{Q}{\delta}\right)$ (continuous case).

If Q is the benefit premium, then we have

1. Fully Discrete Whole Life - $Q = P_x = \frac{A_x}{\ddot{a}_x}$

$$\left(1+\frac{Q}{d}\right) = \left(1+\frac{P_x}{d}\right) = 1+\frac{A_x}{d \cdot \ddot{a}_x} = \frac{d \cdot \ddot{a}_x + A_x}{d \cdot \ddot{a}_x} = \frac{1}{d \cdot \ddot{a}_x} = \frac{1}{1-A_x}$$

Observe that $A_x = P_x \cdot \ddot{a}_x$. Also, recall that $A_x = 1 - d \cdot \ddot{a}_x$. Therefore

$P_x \cdot \ddot{a}_x = 1 - d \cdot \ddot{a}_x$ and solving for $\ddot{a}_x$ gives $\ddot{a}_x = \frac{1}{P_x + d}$.

2. Fully Discrete n-year Endowment Insurance - $Q = P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

$$\left(1+\frac{Q}{d}\right) = \left(1+\frac{P_{x:\overline{n}|}}{d}\right) = 1+\frac{A_{x:\overline{n}|}}{d \cdot \ddot{a}_{x:\overline{n}|}} = \frac{d \cdot \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}}{d \cdot \ddot{a}_{x:\overline{n}|}} = \frac{1}{d \cdot \ddot{a}_{x:\overline{n}|}} = \frac{1}{1-A_{x:\overline{n}|}}$$

Observe that $A_{x:\overline{n}|} = P_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{n}|}$. Also, recall that $A_{x:\overline{n}|} = 1 - d \cdot \ddot{a}_{x:\overline{n}|}$.

So $P_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{n}|} = 1 - d \cdot \ddot{a}_{x:\overline{n}|}$ and solving for $\ddot{a}_{x:\overline{n}|}$ gives $\ddot{a}_{x:\overline{n}|} = \frac{1}{P_{x:\overline{n}|} + d}$.

3. Fully Continuous Whole Life - $Q = \bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}$

$$\left(1+\frac{Q}{\delta}\right) = \left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right) = 1+\frac{\bar{A}_x}{\delta \cdot \bar{a}_x} = \frac{\delta \cdot \bar{a}_x + \bar{A}_x}{\delta \cdot \bar{a}_x} = \frac{1}{\delta \cdot \bar{a}_x} = \frac{1}{1-\bar{A}_x}$$

Observe that $\bar{A}_x = \bar{P}(\bar{A}_x) \cdot \bar{a}_x$. Also, recall that $\bar{A}_x = 1 - \delta \cdot \bar{a}_x$. Therefore

$\bar{P}(\bar{A}_x) \cdot \bar{a}_x = 1 - \delta \cdot \bar{a}_x$ and solving for $\bar{a}_x$ gives $\bar{a}_x = \frac{1}{\bar{P}(\bar{A}_x) + \delta}$.

4. Fully Continuous *n*-year Endowment Insurance - $Q = \bar{P}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$

$$\left(1+\frac{Q}{\delta}\right) = \left(1+\frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\delta}\right) = 1+\frac{\bar{A}_{x:\overline{n}|}}{\delta \cdot \bar{a}_{x:\overline{n}|}} = \frac{\delta \cdot \bar{a}_{x:\overline{n}|} + \bar{A}_{x:\overline{n}|}}{\delta \cdot \bar{a}_{x:\overline{n}|}} = \frac{1}{\delta \cdot \bar{a}_{x:\overline{n}|}} = \frac{1}{1-\bar{A}_{x:\overline{n}|}}$$

Observe that $\bar{A}_{x:\overline{n}|} = \bar{P}(\bar{A}_{x:\overline{n}|}) \cdot \bar{a}_{x:\overline{n}|}$. Also, recall that $\bar{A}_{x:\overline{n}|} = 1 - \delta \cdot \bar{a}_{x:\overline{n}|}$.

So $\bar{P}(\bar{A}_{x:\overline{n}|}) \cdot \bar{a}_{x:\overline{n}|} = 1 - \delta \cdot \bar{a}_{x:\overline{n}|}$ and solving for $\bar{a}_{x:\overline{n}|}$ gives $\bar{a}_{x:\overline{n}|} = \frac{1}{\bar{P}(\bar{A}_{x:\overline{n}|}) + \delta}$.

Prospective Loss At Time t Random Variable

Notation: ${}_tL$ denotes the prospective loss at time t random variable

$${}_tL = PVFBRV_{x+t} - PVFPRV_{x+t} \text{ , where}$$

$PVFBRV_{x+t}$ = PVRV of Future Benefits from age $x + t$, and
$PVFPRV_{x+t}$ = PVRV of Future Premiums from age $x + t$

Note: ${}_0L = L$ = loss-at-issue random variable.

Terminal (End of Year) Reserves

Notation: ${}_tV$ denotes the t^{th} year terminal (EOY) reserves

$${}_tV = E[{}_tL \mid T(x) \geq t] \quad ({}_tV = E[{}_tL \mid K(x) \geq t] \text{ in the discrete case})$$

Prospective Calculation of Reserves:

$${}_tV = APVFB_{x+t} - APVFP_{x+t} \text{ , where}$$

$APVFB_{x+t}$ = APV of Future Benefits from age $x + t$, and
$APVFP_{x+t}$ = APV of Future Premiums from age $x + t$

Benefit reserves means the premiums are the benefit premiums. Benefit reserves can be calculated either prospectively or retrospectively.

Retrospective Calculation of Benefit Reserves: (A more complete discussion of AAV is found later in these notes.)

$${}_tV = AAVPP_{x+t} - AAVPB_{x+t} \text{ , where}$$

$AAVPP_{x+t}$ = AAV of Past Premiums up to age $x + t$, and
$AAVPB_{x+t}$ = AAV of Past Benefits up to age $x + t$

Examples of Prospective Terminal Benefit Reserves and Notation:
(Insurance Benefit = 1)

1. Fully Discrete Whole Life Insurance

$${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$$

2. Fully Discrete *n*-year Pure Endowment

$${}_tV_{x:\overline{n|}}^{\ \ 1} = A_{x+t:\overline{n-t|}}^{\ \ \ 1} - P_{x:\overline{n|}}^{\ \ 1}\,\ddot{a}_{x+t:\overline{n-t|}}$$ (Important Fact: ${}_nV_{x:\overline{n|}}^{\ \ 1} = 1$)

3. Fully Discrete *n*-year Endowment Insurance

$${}_tV_{x:\overline{n|}} = A_{x+t:\overline{n-t|}} - P_{x:\overline{n|}}\ddot{a}_{x+t:\overline{n-t|}} = {}_tV_{x:\overline{n|}}^{1} + {}_tV_{x:\overline{n|}}^{\ \ 1}$$ (Important Fact: ${}_nV_{x:\overline{n|}} = 1$)

4. Fully Discrete *h*-payment Whole Life Insurance

$${}_t^hV_x = \begin{cases} A_{x+t} - {}_hP_x\ddot{a}_{x+t:\overline{h-t|}} \cdots t < h \\ A_{x+t} \cdots t \geq h \end{cases}$$

5. Fully Continuous *n*-year Term Insurance

$${}_t\overline{V}(\overline{A}_{x:\overline{n|}}^{1}) = \overline{A}_{x+t:\overline{n-t|}}^{1} - \overline{P}(\overline{A}_{x:\overline{n|}}^{1}) \cdot \overline{a}_{x+t:\overline{n-t|}}$$

6. Fully Continuous *n*-year Deferred Life Annuity

$${}_t\overline{V}({}_{n|}\overline{a}_x) = \begin{cases} {}_{n-t|}\overline{a}_{x+t} - \overline{P}({}_{n|}\overline{a}_x) \cdot \overline{a}_{x+t:\overline{n-t|}} \cdots t < n \\ \overline{a}_{x+t} \cdots t \geq n \end{cases}$$

7. Semi-continuous Whole Life Insurance

$${}_tV(\overline{A}_x) = \overline{A}_{x+t} - P(\overline{A}_x) \cdot \ddot{a}_{x+t}$$

⋮

You get the idea.

Special Benefit Reserve Formulas
Whole Life and Endowment Insurance

1. $_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x} = (P_{x+t} - P_x)\cdot \ddot{a}_{x+t} = \frac{P_x - P^{\;1}_{x:\overline{t|}}}{P^{\;1}_{x:\overline{t|}}}$

2. $_tV_{x:\overline{n|}} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}} = (P_{x+t:\overline{n-t|}} - P_{x:\overline{n|}})\cdot \ddot{a}_{x+t:\overline{n-t|}}$

3. $_t\bar{V}(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} = [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\cdot \bar{a}_{x+t} = \frac{\bar{P}(\bar{A}_x) - \bar{P}(\bar{A}^{\;1}_{x:\overline{t|}})}{\bar{P}(\bar{A}^{\;1}_{x:\overline{t|}})}$

4. $_t\bar{V}(\bar{A}_{x:\overline{n|}}) = 1 - \frac{\bar{a}_{x+t:\overline{n-t|}}}{\bar{a}_{x:\overline{n|}}} = [\bar{P}(\bar{A}_{x+t:\overline{n-t|}}) - \bar{P}(\bar{A}_{x:\overline{n|}})]\cdot \bar{a}_{x+t:\overline{n-t|}}$

Terminal vs Initial Reserves and Retrospective Actuarial Calculations

Calculate terminal reserves at time *t* by calculating reserves at the end of the t^{th} year. In a prospective calculation, the benefit for the t^{th} year will not be included, but the premium for the $(t+1)^{st}$ year is included.

Calculate initial reserves at time *t* by calculating reserves at the beginning of the $(t+1)^{st}$ year. In a prospective calculation, the benefit for the t^{th} year will not be included and the premium for the $(t+1)^{st}$ year will not be included.

The initial reserves at time *t* will exceed the terminal reserves at time *t* by the amount of premium paid at time *t*.

The retrospective calculation of reserves relies on being able to calculate the Actuarial Accumulated Value (AAV) of contingent payments. For a contingent payment at time *k*, the AAV at time *n* is calculated by actuarially accumulating the APV at time 0 of the contingent payment to time *n*. The APV at time 0 of a contingent payment is the interest discounted value of the expected payment. If a payment *C* is made at time *k* contingent on event *E*, then the APV at time 0 of the payment is $C\cdot \Pr(E)\cdot v^k$. The AAV at time *n* of this payment is $AAV_n = APV_0 \cdot \frac{1}{{}_nE_x} = \frac{C\cdot \Pr(E)\cdot v^k}{v^n\, {}_np_x} = \frac{C\cdot \Pr(E)\cdot (1+i)^{n-k}}{{}_np_x}$.

Special Variance Formulas for
(Conditional) Loss At Time *t* Random Variable
Fully Continuous or Fully Discrete Whole Life or Endowment Insurance

For fully continuous or fully discrete whole life insurance or endowment insurance, the variance of the (conditional) loss at time *t* random variable is

1. Fully Continuous Whole Life Insurance

$$Var\left({}_tL \mid T(x) \geq t\right) = \left(1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2 \cdot \left({}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2\right) = \left(\frac{1}{1-\bar{A}_x}\right)^2 \cdot \left({}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2\right)$$

2. Fully Continuous *n*-year Endowment Insurance ($t < n$)

$$Var\left({}_tL \mid T(x) \geq t\right) = \left(1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\delta}\right)^2 \cdot \left({}^2\bar{A}_{x+t:\overline{n-t}|} - (\bar{A}_{x+t:\overline{n-t}|})^2\right)$$

$$= \left(\frac{1}{1-\bar{A}_{x:\overline{n}|}}\right)^2 \cdot \left({}^2\bar{A}_{x+t:\overline{n-t}|} - (\bar{A}_{x+t:\overline{n-t}|})^2\right)$$

3. Fully Discrete Whole Life Insurance

$$Var\left({}_tL \mid K(x) \geq t\right) = \left(1 + \frac{P_x}{d}\right)^2 \cdot \left({}^2A_{x+t} - (A_{x+t})^2\right) = \left(\frac{1}{1-A_x}\right)^2 \cdot \left({}^2A_{x+t} - (A_{x+t})^2\right)$$

4. Fully Discrete *n*-year Endowment Insurance ($t < n$)

$$Var\left({}_tL \mid K(x) \geq t\right) = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 \cdot \left({}^2A_{x+t::\overline{n-t}|} - (A_{x+t::\overline{n-t}|})^2\right) = \left(\frac{1}{1-A_{x:\overline{n}|}}\right)^2 \cdot \left({}^2A_{x+t::\overline{n-t}|} - (A_{x+t::\overline{n-t}|})^2\right)$$

Recursive Relationship for Insurance Terminal Reserves
(often tested discrete case)

$${}_tV = b_{t+1} \cdot v \cdot q_{x+t} - Q_t + {}_{t+1}V \cdot v \cdot p_{x+t}$$, where

Q_t = premium at time t, and
b_{t+1} = death benefit at time t+1

Recursive Relationship for Variance of (Conditional) Loss At Time *t* Discrete Random Variable

$$Var({}_tL \mid K(x) \geq t) = \left(v\left(b_{t+1} - {}_{t+1}V\right)\right)^2 \cdot p_{x+t} \cdot q_{x+t} + v^2 \cdot p_{x+t} \cdot Var({}_{t+1}L \mid K(x) \geq t+1)$$

where b_{t+1} = death benefit at time t+1

Approximating Benefit Reserves at Fractional Durations
($0 < s < 1$) – discrete case

$${}_{t+s}V = (1-s)({}_tV + Q_t) + s({}_{t+1}V) = (1-s)({}_tV) + s({}_{t+1}V) + (1-s)(Q_t)$$

where Q_t = premium at time t

Notes:
1. Use linear approximation between the initial and terminal reserves for year t
2. $(1-s)(Q_t)$ is called the unearned premium

Expense Augmented Models

Unless otherwise stated, assume expenses are paid BOY

Exception: Settlement expenses are paid at the time benefit is paid

Replace "benefits" by "benefits plus expenses" and replace "premiums" by "expense loaded premiums" in the random variable expressions discussed above. All formulas use the same concept as in the non-expense model. The following illustrates this.

Recursion Relation for Expense Augmented Terminal Reserves

$${}_tV_{ea} = (b_{t+1} + se) \cdot v \cdot q_{x+t} + E_t - Q_t + {}_{t+1}V_{ea} \cdot v \cdot p_{x+t}$$, where

${}_tV_{ea}$ = expense augmented terminal reserves at time t
se = settlement expenses
E_t = BOY expenses paid at time t
Q_t = expense loaded premium at time t (usually more in 1st year)

Comments on Expense Loaded Premiums

1. Generally, part of the expense loaded premium will depend on the face amount of the policy. The amount of the expense loaded premium that does *not* depend on the face amount of the policy is called the **policy fee**.

2. Letting Q_t denote the expense loaded premium at time t, and letting P_t denote the benefit premium at time t, then $e_t = Q_t - P_t$ is called the expense loading at time t.

3. The expense loaded premium pays expenses, but not any profit. The **contract premium** is charged in order to expect a profit.

Separating Benefits and Expenses in an Expense Augmented Model

${}_tV_{ea} = {}_tV_{ben} + {}_tV_{\exp}$, where
${}_tV_{ben}$ = reserves as before in the non-expense model (using the premiums Q_t)
${}_tV_{\exp}$ = expense reserves
$= APVF(\text{expenses})_{x+t} - APVF(\text{expense loadings})_{x+t}$

Multiple Decrement Models

Two random variables:

$T(x)$ – future lifetime of (x) random variable

J – mode of decrement random variable

Notation:

${}_t q_x^{(j)}$ = Pr((x) departs within t years by decrement j)

${}_t p_x^{(j)}$ has no meaning

${}_t q_x^{(\tau)}$ = Pr((x) departs within t years) = $\sum_j {}_t q_x^{(j)}$

${}_t p_x^{(\tau)}$ = Pr((x) survives all decrements for t years) = $1 - {}_t q_x^{(\tau)}$

$\mu_x^{(j)}(t)$ = force of mortality due to decrement j

$\mu_x^{(\tau)}(t)$ = total force of mortality = $\sum_j \mu_x^{(j)}(t)$

Multiple Decrement Service Table Notation

$l_x^{(\tau)}$ = the total number that have survived all decrements to age x

$l_x^{(j)}$ = the total number at age x that will eventually depart by cause j

${}_n d_x^{(\tau)}$ = the total number departing in next n years

$${}_\infty d_x^{(\tau)} = l_x^{(\tau)}$$

${}_n d_x^{(j)}$ = the total number departing in next n years by cause j

$${}_\infty d_x^{(j)} = l_x^{(j)}$$

Multiple Decrement Model Facts:

1. $f_{T,J}(t,j) = {}_tp_x^{(\tau)} \cdot \mu_x^{(j)}(t)$

2. $f_T(t) = \sum_J {}_tp_x^{(\tau)} \cdot \mu_x^{(j)}(t) = {}_tp_x^{(\tau)} \cdot \mu_x^{(\tau)}(t)$ and $f_J(j) = \int_0^\infty {}_tp_x^{(\tau)} \cdot \mu_x^{(j)}(t)dt = {}_\infty q_x^{(j)}$

3. ${}_tq_x^{(\tau)} = \int_0^t {}_sp_x^{(\tau)} \cdot \mu_x^{(\tau)}(s)ds = \frac{{}_td_x^{(\tau)}}{l_x^{(\tau)}}$

4. ${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \cdot \mu_x^{(j)}(s)ds = \frac{{}_td_x^{(j)}}{l_x^{(\tau)}}$

5. ${}_{t|u}q_x^{(\tau)} = \begin{cases} {}_tp_x^{(\tau)} - {}_{t+u}p_x^{(\tau)} \\ {}_{t+u}q_x^{(\tau)} - {}_tq_x^{(\tau)} \\ {}_tp_x^{(\tau)} \cdot {}_uq_{x+t}^{(\tau)} \end{cases} = \frac{{}_ud_{x+t}^{(\tau)}}{l_x^{(\tau)}}$

6. ${}_{t|u}q_x^{(j)} = {}_tp_x^{(\tau)} \cdot {}_uq_{x+t}^{(j)} = \frac{{}_ud_{x+t}^{(j)}}{l_x^{(\tau)}}$

7. $E[T] = \overset{0}{e}{}_x^{(\tau)} = \int_0^\infty {}_tp_x^{(\tau)}dt$ = expected time until decrement

8. $\Pr(K(x) = k) = \Pr(k \le T(x) < k+1) = {}_{k|}q_x^{(\tau)}$

9. $\Pr(J = j \mid T = t) = f_{J|T}(j \mid t) = \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)}$

10. $\Pr(J = j \mid T \le t) = \frac{{}_tq_x^{(j)}}{{}_tq_x^{(\tau)}}$

11. $f_{T|J}(t \mid j) = \frac{f_{T,J}(t,j)}{f_J(j)}$

12. $E[T \mid J = j] = \int_0^\infty t \cdot \frac{f_{T,J}(t,j)}{f_J(j)}dt$ = expected time until departure, given the cause is by decrement j

Associated Single Decrement Tables (Absolute Rates of Decrement)
(Associated single decrement events are independent)

Probability Formulas:

$$ {}_t p_x'^{(j)} = \exp(-\int_0^t \mu_x^{(j)}(s)ds) \Rightarrow \mu_x^{(j)}(t) = \frac{-\frac{d}{dt}[{}_t p_x'^{(j)}]}{{}_t p_x'^{(j)}} $$

$$ {}_t q_x'^{(j)} = 1 - {}_t p_x'^{(j)} = \int_0^t {}_s p_x'^{(j)} \cdot \mu_x^{(j)}(s)ds $$

$$ {}_t p_x^{(\tau)} = \prod_j {}_t p_x'^{(j)} $$

Calculating Total Probabilities:

If given associated single decrement probabilities (primes) then calculate total probabilities by

$$ {}_t p_x^{(\tau)} = \prod_j {}_t p_x'^{(j)} \text{ and } {}_t q_x^{(\tau)} = 1 - {}_t p_x^{(\tau)} $$

If given multiple decrement probabilities (no primes) then calculate total probabilities by

$$ {}_t q_x^{(\tau)} = \sum_j {}_t q_x^{(j)} \text{ and } {}_t p_x^{(\tau)} = 1 - {}_t q_x^{(\tau)} $$

Relating Multiple Decrement Probabilities (no primes) to Associated Single Decrement Probabilities (primes)

Two Cases:

Case 1: MUDD
(${}_t q_x^{(j)} = t \cdot q_x^{(j)}$; UDD assumption in the multiple decrement model)

Important Formula: ${}_s p_x'^{(j)} = ({}_s p_x^{(\tau)})^{q_x^{(j)} / q_x^{(\tau)}}$

Case 2: SUDD
(${}_t q_x'^{(j)} = t \cdot q_x'^{(j)}$; UDD assumption in the associated single decrement model)

Important Formulas: $q_x^{(1)} = q_x'^{(1)} \cdot \left(1 - \frac{1}{2} q_x'^{(2)}\right)$ (2 decrement case)

$$q_x^{(1)} = q_x'^{(1)} \cdot \left(1 - \frac{1}{2}(q_x'^{(2)} + q_x'^{(3)}) + \frac{1}{3} q_x'^{(2)} \cdot q_x'^{(3)}\right) \text{ (3 decrement case)}$$

Interchange the roles of the superscripts to get probabilities of departing by causes other than cause 1. For example, interchanging the roles of 1 and 2 in the 3 decrement case gives us the probability

$$q_x^{(2)} = q_x'^{(2)} \cdot \left(1 - \frac{1}{2}(q_x'^{(1)} + q_x'^{(3)}) + \frac{1}{3} q_x'^{(1)} \cdot q_x'^{(3)}\right)$$

It is unlikely that you will see more than 3 decrements.

Discrete Decrements

If some decrements happen at a fixed point in time, then calculate 1-year mortality probabilities in the associated single decrement table by using the formula $q_x'^{(j)} = \frac{d_x^{(j)}}{NAR_x^{(j)}}$, where $NAR_x^{(j)}$ is the number at risk for decrement *j*, at age *x*. Note that $q_x^{(j)} = \frac{d_x^{(j)}}{l_x}$ always. Also, Venn Diagrams may be used in the SUDD or SUDD/Discrete Decrement case. We will illustrate with examples.

Asset Shares

This concept is based on a double decrement model, death and withdrawal. The death benefit at time h is denoted by ${}_{h}DB$. If withdrawal occurs at time h, then a cash value, denoted ${}_{h}CV$, is paid at that time. We let ${}_{h}AS$ denote the asset share at time h. Assume ${}_{0}AS = 0$. There are two main formulas.

Recursion Formula:

$$ {}_{h}AS + G(1 - c_h) - E_h = ({}_{h+1}DB + {}_{h+1}se) \cdot v \cdot q_{x+h}^{(d)} + ({}_{h+1}CV + {}_{h+1}se) \cdot v \cdot q_{x+h}^{(w)} + v \cdot p_{x+h}^{(\tau)} \cdot {}_{h+1}AS $$

where G = contract premium
c_h = percentage of contract premium expense at time h
E_h = other non-settlement expenses at time h
${}_{h+1}se$ = settlement expenses at time $h+1$
$q_{x+h}^{(d)}$ = probability that $(x+h)$ dies by age $x+h+1$
$q_{x+h}^{(w)}$ = probability that $(x+h)$ withdraws by age $x+h+1$