§51. Homotopy of Paths

General note: every problem here uses Theorem 18.2, mostly part (c). You should probably be on good terms with this Theorem.

1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h \simeq k' \circ h'$.

   Since $h \simeq h'$, we have a homotopy
   $$H : X \times I \to Y \quad \text{with } H(x,0) = h(x), H(x,1) = h'(x),$$
   and since $k \simeq k'$, we have a homotopy
   $$K : Y \times I \to Z \quad \text{with } K(x,0) = k(x), K(x,1) = k'(x).$$
   Then define $F : X \times I \to Z$ by
   $$F(x,t) = \begin{cases} k(H(x,2t)), & 0 \le t \le \frac{1}{2} \\ K(h'(x), 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$
   Then check that F does what it oughta do, at different times t:[1]
   $$\begin{array}{ll} t = 0 & F(x,0) = k(H(x,0)) = k(h(x)) = k \circ h(x) \\ t = \frac{1}{2} & F\left(x, \frac{1}{2}\right) = k(H(x,1)) = k(h'(x)) = K(h'(x), 0) \\ t = 1 & F(x,1) = K(h'(x),1) = k'(h'(x)) = k' \circ h'(x). \end{array}$$

2. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y. Let $I = [0, 1]$.

   (a) Show that for any X, the set $[X, I]$ has a single element.

   Define $\tilde{\varphi} : X \to I$ to be the zero map: $\tilde{\varphi}(x) = 0, \forall x \in X$. Let $\varphi : X \to I$ by any continuous map. We show $\varphi \simeq \tilde{\varphi}$: define $\Phi : X \times I \to I$ by
   $$\Phi(x,t) = (1-t)\varphi(x).$$
   This is evidently a homotopy $\varphi \simeq \tilde{\varphi}$.
   Since $\simeq$ is an equivalence relation, this shows all maps $\varphi : X \to I$ are equivalent under $\simeq$, i.e., there is only one equivalence class.

   (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

   Pick any point $p \in Y$ and define $\tilde{\varphi} : I \to Y$ by $\tilde{\varphi}(x) = p$, so $\tilde{\varphi}$ is the constant map at p. Now let $\varphi : I \to Y$ be arbitrary; we will again show $\varphi \simeq \tilde{\varphi}$. Denote $\varphi(0) = a$ and $\varphi(1) = b$.

[1] Here, and elsewhere, we are actually using the Pasting Lemma (Thm 18.3) to ensure this piecewise-defined function is actually continuous. This is justified by the middle calculation, for $t = \frac{1}{2}$.

Since Y is path-connected, there is a path $\gamma : I \to Y$ with $\gamma(0) = a = \varphi(0)$ and $\gamma(1) = p$. Now define a homotopy $\Phi : I \times I \to Y$ by

$$\Phi(x,t) = \begin{cases} \varphi\left((1-2t)x\right), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that Φ does not depend on x once $t \ge \frac{1}{2}$! This is because Φ is the constant map to the point $\gamma(2t-1)$ from this point onwards. Φ has the effect of shrinking the image of φ to a point while $0 \le t \le \frac{1}{2}$, then moving that point to p along $\gamma(I)$ while $\frac{1}{2} \le t \le 1$.

Check that Φ does what it oughta do:

$$\begin{array}{ll} t=0 & \Phi(x,0) = \varphi(x) \\ t = \frac{1}{2} & \Phi\left(x, \frac{1}{2}\right) = \varphi(0) = a = \gamma(0) \\ t=1 & \Phi(x,1) = \gamma(1) = p. \end{array}$$

3. A space X is said to be contractible if the identity map $i_X : X \to X$ is nulhomotopic.

   (a) Show that I and $\mathbb{R}$ are contractible.

   Define the constant map $\tilde{\varphi}(x) = 0$, for either space. Then we define the homotopy $H : X \times I \to X$ by

   $$H(x,t) = (1-t) \cdot id_X(x) = (1-t)x.$$

   This polynomial in x and t is clearly continuous, and

   $$\begin{array}{ll} t=0 & H(x,0) = id_X(x) \\ t=1 & H(x,1) = 0. \end{array}$$

   (b) Show that a contractible space is path connected.

   Let X be a contractible space. Then there is a homotopy H between id_X and some constant map; call it f. So $f(x) = p, \forall x \in X$ and $H : id_X \simeq f$, i.e.,

   $$H : X \times I \to X \quad \text{with } H(x,0) = id_X(x) \text{ and } H(x,1) = f(x) = p.$$

   To show X is path connected, we fix any two points $y, z \in X$ and construct a path between them. Note that $H(y,t)$ is a path from y to p and $H(z,t)$ is a path from z to p (recall that y, z are *fixed*). Thus, define a path by

   $$\gamma(t) = \begin{cases} H(y, 2t), & 0 \le t \le \frac{1}{2} \\ H(z, 2-2t), & \frac{1}{2} \le t \le 1. \end{cases}$$

   And check it:

   $$\begin{array}{ll} t=0 & \gamma(t) = H(y,0) = id_X(y) = y \\ t = \frac{1}{2} & \gamma(t) = H(y,1) = p = H(z,1) \end{array}$$

$$t = 1 \qquad \gamma(t) = H(z,0) = id_X(z) = z.$$

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

Just as above, if Y is contractible, then we have a homotopy H between id_Y and some constant map; call it f. So $f(y) = p, \forall y \in Y$ and $H : id_Y \simeq f$, i.e.,

$$H : Y \times I \to Y \quad \text{ with } H(y, 0) = id_Y(y) \text{ and } H(y, 1) = f(y) = p.$$

Take any arbitrary map $\varphi : X \to Y$ and define

$$\Phi(x,t) = H(\varphi(x), t)).$$

Then

$$\begin{aligned} t = 0 \qquad & \Phi(x,0) = H(\varphi(x), 0)) = \varphi(x) \\ t = 1 \qquad & \Phi(x,1) = H(\varphi(x), 1)) = f(x) = p. \end{aligned}$$

So every map $\varphi : X \to Y$ is homotopic to the constant map at p.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Let X be a contractible space. Then there is a homotopy H between id_X and some constant map; call it f. So $f(x) = p, \forall x \in X$ and $H : id_X \simeq f$, i.e.,

$$H : X \times I \to X \quad \text{ with } H(x, 0) = id_X(x) \text{ and } H(x, 1) = f(x) = p.$$

Pick some other point $q \in Y$ and define a constant map $\tilde{\varphi} : Y \to Y$ by $\tilde{\varphi}(y) = q, \forall y \in Y$. Take any arbitrary map $\varphi : X \to Y$. We will show $\varphi \simeq \tilde{\varphi}$, so that all maps from X to Y are homotopic to $\tilde{\varphi}$.
The plan is: use the contractibility of X to shrink it to a point (at p), then use the path-connectedness of Y to move $\varphi(p)$ (which is in Y) to $q \in Y$. So we need a path γ from $\varphi(p)$ to q. By the path-connectedness of Y we have one:

$$\gamma : I \to Y, \qquad \text{with } \gamma(0) = \varphi(p), \gamma(1) = q.$$

Now we can set up the requisite homotopy.

$$\Phi(x,t) = \begin{cases} \varphi(H(x,2t)), & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

And check it:

$$\begin{aligned} t &= 0 & \Phi(x,0) &= \varphi(id_X(x)) = \varphi(x) \\ t &= \tfrac{1}{2} & \Phi\left(x,\tfrac{1}{2}\right) &= \varphi(p) = \gamma(0) \\ t &= 1 & \Phi(x,1) &= \gamma(1) = q = \tilde{\varphi}(x). \end{aligned}$$

§52. The Fundamental Group

1. A subset A of $\mathbb{R}^n$ is *star convex* iff for some point $a_0 \in A$, all the line segments joining a_0 to other points of A lie in A, i.e., $(1-\lambda)a + \lambda a_0 \in A, \forall \lambda \in (0,1)$.

   (a) Find a star convex set that is not convex.

   A six-pointed star like the Star of David, or a pentacle will work if you let a_0 be the center. (Hence the name "star convex".)
   The set $\{(x,y) \vdots x = 0 \text{ or } y = 0\} \subseteq \mathbb{R}^2$ is star convex with respect to the origin. Or let $I^2 = I \times I \subseteq \mathbb{R}^2$ and let $X = \{(x,y) \vdots y = 0\} \subseteq \mathbb{R}^2$. Then $A = I^2 \cup X$ is a star convex subset of $\mathbb{R}^2$ which is not convex. The convex hull of A (smallest convex set containing A, or intersection of all convex sets containing A) is

   $$conv(A) = \{(x,y) \vdots 0 \leq y < 1\} \cup X.$$

   (b) Show that if A is star convex, A is simply connected.

   Let $a \in A$ be a point satisfying the definition of star convexity. Then $H : A \times I \to A$ by

   $$H(x,t) = (1-t)x + ta$$

   shows A is contractible (via the straight line homotopy). Thus, $[X, A]$ consists of single element, by §51, Exercise 3(c), for any space X. In particular, this is true for $[S^1, A]$. Now let $[S^1, A]_a \subseteq [S^1, A]$ be those maps which send at least one point of S^1 to a. Then $[S^1, A]_a$ also consists of a single element. But $[S^1, A]_a = \pi_1(A, x_0)$! So $\pi_1(A, x_0)$ is trivial.

2. Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

   We show $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$ by showing that $\hat{\gamma}([f]) = (\hat{\beta} \circ \hat{\alpha})([f])$, for every path f.

   $$\begin{aligned}
   \hat{\gamma}([f]) &= [\bar{\gamma}] * [f] * [\gamma] & \\
   &= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] & \text{def of } \gamma \\
   &= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha * \beta] & \text{def of the reverse } \bar{\alpha} \\
   &= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta] & \text{def of } *, \text{ p.326} \\
   &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] & \text{def of } \hat{\alpha} \\
   &= \hat{\beta}(\hat{\alpha}([f])) & \text{def of } \hat{\beta} \\
   &= (\hat{\beta} \circ \hat{\alpha})([f]) &
   \end{aligned}$$

   Note that the reverse of $\alpha * \beta$ (α followed by β) is the reverse of β followed by the reverse of α, in the third line above.

3. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian iff for every pair α, β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

($\Rightarrow$) Suppose $\pi_1(X, x_0)$ is abelian, and let α, β be paths from x_0 to x_1. Since $\hat{\alpha}$ and $\hat{\beta}$ are both homomorphisms from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$, we need to prove that they both send a loop $f \in \pi_1(X, x_0)$ to the same loop in $\pi_1(X, x_1)$. Note that $\pi_1(X, x_1)$ must also be abelian, since $\hat{\alpha}$ is an isomorphism, by Cor 52.2 (we are given that X is path connected). Now we show $\hat{\alpha} = \hat{\beta}$.

$$\begin{aligned}
\hat{\alpha}([f]) &= [\bar{\alpha}] * [f] * [\alpha] && \text{def of } \hat{\alpha} \\
&= [\bar{\alpha}] * [\alpha] * [f] && \pi_1(X, x_1) \text{ is abelian} \\
&= [e_{x_1}] * [f] && \text{Thm 51.2(3)} \\
&= [f] * [e_{x_1}] && \pi_1(X, x_1) \text{ is abelian} \\
&= [f] && \text{Thm 51.2(2)}
\end{aligned}$$

Similarly, $\bar{\beta}([f]) = [f]$. So we have actually proven that $\hat{\alpha}$ and $\hat{\beta}$ are the identity, i.e., that if X is path connected, it must also be simply connected. $\lightning$

Of course, this is all completely wrong! Why? It makes no sense: $[\alpha]$ is not an element of $\pi_1(X, x_1)$ because it isn't even a loop! Therefore, the fact that $\pi_1(X, x_0)$ is abelian doesn't apply. The expression $[\alpha] * [f]$, which appears in line 2 isn't even defined, since α ends at x_1, and f ends at x_0.

Try again:

$$\begin{aligned}
[f] * [\alpha] &= [f] * [\alpha] * [e_{x_1}] && \text{Thm 51.2(2)} \\
&= [f] * [\alpha] * \left[\bar{\beta}\right] * [\beta] && \text{Thm 51.2(3)} \\
&= [f] * \left[\alpha * \bar{\beta}\right] * [\beta] && \text{def of } *, \text{ p.326} \\
&= \left[\alpha * \bar{\beta}\right] * [f] * [\beta] && \pi_1(X, x_0) \text{ is abelian} \\
&= [\alpha] * \left[\bar{\beta}\right] * [f] * [\beta] && \text{def of } *, \text{ p.326} \\
[\bar{\alpha}] * [f] * [\alpha] &= [\bar{\alpha}] * [\alpha] * \left[\bar{\beta}\right] * [f] * [\beta] && \text{multiply both sides on the left} \\
[\bar{\alpha}] * [f] * [\alpha] &= [e_{x_1}] * \left[\bar{\beta}\right] * [f] * [\beta] && \text{Thm 51.2(3)} \\
[\bar{\alpha}] * [f] * [\alpha] &= \left[\bar{\beta}\right] * [f] * [\beta] && \text{Thm 51.2(2) } \textit{and} \text{ Thm 51.2(2)} \\
\hat{\alpha}([f]) &= \hat{\beta}([f]) && \text{def of } \hat{\alpha}, \hat{\beta}
\end{aligned}$$

Whew! Note that the fourth equality above is justified because $\left[\alpha * \bar{\beta}\right]$ *is* a loop, so we avoid the problem of the other (incorrect) solution above.

($\Leftarrow$) Now suppose that $\hat{\alpha} = \hat{\beta}$ for every pair α, β of paths from x_0 to x_1. We must show $\pi_1(X, x_0)$ is abelian, so pick $f, g \in \pi_1(X, x_0)$ and show $[f] * [g] = [g] * [f]$. Now we have the freedom to choose our α and β, so define

$$\alpha = f, \beta = f * g.$$

Then compute:

$$\begin{aligned}
\hat{\alpha}([f]) &= \widehat{f}([f]) && \alpha = f \\
&= [\bar{f}] * [f] * [f] && \text{def of } \hat{f} \\
&= [e_{x_0}] * [f] && \text{Thm 51.2(3)}, e_{x_0} = e_{x_1} \\
&= [f] && \text{Thm 51.2(2)}
\end{aligned}$$

and

$$\begin{aligned}
\hat{\beta}([f]) &= \widehat{f * g}([f]) && \beta = f * g \\
&= [\overline{f * g}] * [f] * [f * g] && \text{def of } \hat{f} \\
&= [\overline{g}] * [\overline{f}] * [f] * [f] * [g] && \text{see Ex 2, at the end} \\
&= [\overline{g}] * [e_{x_0}] * [f] * [g] && \text{Thm 51.2(3)}, e_{x_0} = e_{x_1} \\
&= [\overline{g}] * [f] * [g] && \text{Thm 51.2(2).}
\end{aligned}$$

Since $\hat{\alpha} = \hat{\beta}$, we know $\hat{\alpha}([f]) = \hat{\beta}([f])$, and so

$$\begin{aligned}
[f] &= [\overline{g}] * [f] * [g] \\
[g] * [f] &= [g] * [\overline{g}] * [f] * [g] \\
[g] * [f] &= [f] * [g],
\end{aligned}$$

using Thm 51.2(2,3) to cancel the g's. So $\pi_1(X, x_0)$ is abelian.

There is actually a much simpler way to do each direction of this proof. Can you find it? Hint: remember the old tricks of "adding 0" or "multiplying by 1" for the forward direction, and choose a more clever α, β for the backward direction.

4. Let $A \subseteq X$; suppose $r : X \longrightarrow A$ is a *retraction*, i.e., a continuous map such that $r(a) = a$ for each $a \in A$. If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Let $f : I \longrightarrow A$ be a loop in A, based at x_0. We must find a loop g in X, based at x_0, such that $r_*([g]) = [r \circ g] = [f]$. Because f is also a loop in X, based at x_0, we can let $g = f$. Then

$$r \circ f(t) = r(f(t)) = f(t), \quad \forall t \in I,$$

since $f(t) \in A$ and r is the identity on A.

5. Let $A \in \mathbb{R}^n$ and let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism.

Let $[f] \in \pi_1(A, a_0)$ so that $f : I \to A$ is a loop at a_0, and consider $h_*([f]) = [h \circ f] \in \pi_1(Y, y_0)$. The extendability of h says that we can find a continuous map $g : (\mathbb{R}^n, a_0) \to (Y, y_0)$ such that $g(a) = a$ whenever $a \in A$. Then

$$(g \circ f)(t) = g(f(t)) = h(f(t)) = (h \circ f)(t), \quad \forall t \in I,$$

since $f(t) \in A$ and $g = h$ on A.

Strategy: if $h \circ f$ were nulhomotopic, then we would have

$$h_*([f]) = [h \circ f] = [e_{y_0}], \quad \forall f,$$

so that h_* is trivial. So we show $h \circ f$ to be nulhomotopic.

Define $\tilde{\varphi} : I \to Y$ by $\tilde{\varphi}(t) = y_0, \forall t$. Then define $\Phi : \mathbb{R}^n \times I \to Y$ by

$$\Phi(x,t) = g((1-t)f(x) + ta_0),$$

and check

$$\begin{array}{ll} t = 0 & \Phi(x,0) = g(f(x)) = g \circ f(x) = h \circ f(x) \\ t = 1 & \Phi(x,1) = g(a_0) = y_0 = \tilde{\varphi}(x). \end{array}$$

Note: we don't know that $h((1-t)f(x) + ta_0)$ is defined, but since h is extendible to a continuous map on all of $\mathbb{R}^n$, we konw that $g((1-t)f(x) + ta_0)$ *is* defined, since $(1-t)f(x) + ta_0$ is just some point between $f(x) \in A$ and a_0.

6. Show that if X is path connected, the homomorphism induced by a continuous map is independent of a base point, up to isomorphisms of the groups involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha},$$

so that the diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha} \big\downarrow & & \big\downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

We need to see that these two operations do the same thing to any $[f] \in \pi_1(X, x_0)$, so let f be a loop in X based at x_0. On the one hand, we have

$$\begin{aligned} \hat{\beta} \circ (h_{x_0})_*([f]) &= \widehat{h \circ \alpha} \circ (h_{x_0})_*([f]) && \text{def of } \beta \\ &= \widehat{h \circ \alpha}([h \circ f]) && \text{def of } h_* \\ &= \overline{[h \circ \alpha]} * [h \circ f] * [h \circ \alpha] && \text{def of } \hat{\beta} \\ &= \overline{[h \circ \alpha]} * [h \circ f] * [h \circ \alpha] && \text{def of } \hat{\beta}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(h_{x_1})_* \circ \hat{\alpha}([f]) &= (h_{x_1})_* ([\bar{\alpha}] * [f] * [\alpha]) && \text{def of } \hat{\alpha} \\
&= (h_{x_1})_* ([\bar{\alpha} * f * \alpha]) && \text{def of } * \\
&= [h \circ (\bar{\alpha} * f * \alpha)] && \text{def of } h_* \\
&= [(h \circ \bar{\alpha}) * (h \circ f) * (h \circ \alpha))] && k \circ (f * g) = (k \circ f) * (k \circ g) \\
&= [h \circ \bar{\alpha}] * [h \circ f] * [h \circ \alpha] && \text{def of } *.
\end{aligned}$$

So we still need $\overline{[h \circ \alpha]} = [h \circ \bar{\alpha}]$. But this is true:

$$\overline{h \circ \alpha}(t) = (h \circ \alpha)(1 - t) = h(\alpha(1 - t)) = h(\bar{\alpha}(t)) = h \circ \bar{\alpha}(t).$$

§53. Covering Spaces

1. Let Y have the discrete topology. Show that if $p : X \times Y \to X$ is projection on the first coordinate, then p is a covering map.

   It is clear that p is continuous and surjective (if you have doubts, read pp. 107–110). Pick $x \in X$ and let U be a neighbourhood of x. We will show that U is evenly covered by p; that is, that $p^{-1}(U)$ can be written as a union of disjoint sets $V_\alpha \subseteq X \times Y$ such that for each α, $p\big|_\alpha : V_\alpha \to U$ is a homeomorphism.

   Since Y is discrete, $\{y\}$ is open in Y and so $U \times \{y\}$ is open in $X \times Y$ by the def of product topology. Then we show

$$p^{-1}(U) = \bigsqcup_{y \in Y} U \times \{y\}.$$

   Note: this union is disjoint because

$$\begin{aligned}(x, y) \in U \times \{y_1\} \cap U \times \{y_2\} \quad &\Longrightarrow \quad y = y_1 = y_2 \\ &\Longrightarrow \quad U \times \{y_1\} = U \times \{y_2\}.\end{aligned}$$

   Let $x \in p^{-1}(U)$

2. Let $p : E \to B$ be continuous and surjective. Suppose that U is an open set of B that is evenly covered by p. Show that if U is connected, then the partition of $p^{-1}(U)$ into slices is unique.

   Suppose we have two partitions of $p^{-1}(U)$ into slices:

$$\mathcal{A} = \{V_\alpha\}_{\alpha \in A} \text{ and } \mathcal{B} = \{V_\beta\}_{\beta \in B}.$$

   Fix $b \in B$. Then for any α, we can find the unique point (by homeomorphism) $b_\alpha \in V_\alpha$ such that $p(b_\alpha) = b$, and for any β, we can find the unique point $b_\beta \in V_\beta$ such that $p(b_\beta) = b$. Note that every V_α, V_β is connected, by homeomorphism with U. We will show that there is a bijection between these partitions; i.e., $\mathcal{A}$ is actually just a reindexing of $\mathcal{B}$.

   Fix α_0. Define $f : A \to B$ as follows: find the unique β_0 such that $b_{\alpha_0} \in V_{\beta_0}$ and define $f(\alpha_0) = \beta_0$. There will be such a V_{β_0}, because $E = \cup V_\beta$, and that V_{β_0} will be unique by the disjointness of the V_β.

   Now to see that this is a bijection. For $b_\beta \in V_\beta \subseteq E$, $\exists \alpha$ such that $b_\beta \in V_\alpha$ because $\bigcup V_\alpha$ contains E. This shows surjectivity. For injectivity, note that

$$b_\beta \in V_{\alpha_1} \cap V_{\alpha_1} \quad \Longrightarrow \quad \alpha_1 = \alpha_2$$

   by the disjointness of the partition.

3. Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$, i.e., E is a k-fold covering of B.

   Since $|p^{-1}(b_0)| = k$, we can find U such that

$$p^{-1}(U) = \bigsqcup_{i=1}^{k} V_i \qquad \text{and} \qquad p\big|_{V_i} = p_i : V_i \to U \text{ is a homeomorphism}, \forall i.$$

   We assume that $\exists b_1$ such that $|p^{-1}(b_1)| = j \neq k$, and we will contradict the fact that B is connected. Define

$$C = \{b : |p^{-1}(b)| = k\} \quad \text{and} \quad D = \{b : |p^{-1}(b)| \neq k\}.$$

   We have $b_0 \in C$, so $C \neq \varnothing$, and $b_1 \in D$, so $D \neq \varnothing$. Also, it is clear that $C \cap D = \varnothing$ and $C \cup D = B$. So it just remains to show C, D are open.

   For $b \in C$, we can find U_b such that $p^{-1}(U_b) = \bigsqcup_{i=1}^{k} V_i$, where the V_i are open. Thus for $x \in U_b$, $|p^{-1}(x)| = k$. Hence $b \in U_b \subseteq C$ shows C is open. Similarly, D is open. This gives C, D as a disconnection of B. $\lightning$ Hence no such b_1 exists.

4. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.

   As r is a covering map, pick $z \in Z$ and find an open neighbourhood Z of z such that $p^{-1}(Z) = \bigsqcup_{i=1}^{n} V_i$ where V_i is open in Y and $r\big|_{V_i}: V_i \to Z$ is a homeomorphism. Define v_i to be the single element of $p^{-1}(z) \cap V_i$, for each i. As q is a covering map, we can find an open neighbourhood U_i of v_i such that $q^{-1}(U_i) = \bigsqcup_{\alpha} A_{i_\alpha}$ where A_{i_α} is open in X and $q\big|_{A_{i_\alpha}}: A_{i_\alpha} \to U_i$ is a homeomorphism.

   With so many sets, we need to find some common ground, so define

$$C = \bigcap_{i=1}^{n} r(U_i \cap V_i).$$

   Each $U_i \cap V_i$ is a neighbourhood of v_i in Y which is evenly covered by q, and has $r(U_i \cap V_i) \subseteq Z$. So C is evenly covered by r (each slice is a $U_i \cap V_i$). Most importantly, the fact that C is a *finite* intersection guarantees that C is open. We next define

$$D_{i_\alpha} = q^{-1}(U_i \cap V_i) \cap A_{i_\alpha},$$

   which are also open. Now C is a neighbourhood of $z \in Z$ for which

$$p^{-1}(C) = \bigsqcup_{i,\alpha} D_{i_\alpha} \quad \text{and} \quad p\big|_{D_{i_\alpha}} = (r \circ q)\big|_{D_{i_\alpha}} \text{ is a homeomorphism.}$$

   Also, the D_{i_α} are disjoint because the A_{i_α} are.

5. Show that the map $p : S^1 \to S^1$ given by $p(z) = z^2$ is a covering map. Generalize to the map $p(z) = z^n$.

Here we consider $S^1 \subseteq \mathbb{C}$, so that for $z \in S^1$ we may write

$$z = e^{i\theta} \qquad \text{and} \qquad p(z) = z^2 = e^{2i\theta}.$$

For $z \in S^1$, let $z = e^{i\theta}$ so $\theta = \arg(z) \in [0, 2\pi)$. Let U be the image of $\left(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right)$ under the map $\theta \mapsto e^{i\theta}$ so that U is the open semicircle centered at z. Then $p^{-1}(U)$ consists of the "quarter circle" centered at z

$$V_1 = \exp\left(\theta - \tfrac{\pi}{4}, \theta + \tfrac{\pi}{4}\right)$$

and the "quarter circle" centered at $-z$

$$V_2 = \exp\left(-\theta + \tfrac{\pi}{4}, -\theta - \tfrac{\pi}{4}\right).$$

Clearly, $U = V_1 \sqcup V_2$ and U is homeomorphic to each of V_1, V_2 by $p(z) = z^2$. Since z has the requisite neighbourhood, we are done.

For $p(z) = z^n$, use the same U. You will get $p^{-1}(U) = \bigsqcup_{i=1}^{n} V_i$, where each V_i goes $\frac{1}{2n}$ of the way around the circle S^1.

Bonus Problem: If $p : E \to B$ is a covering map, show that $p^{-1}(b) \subseteq E$ has the discrete topology, for any $b \in B$.

Consider the topology that $p^{-1}(b)$ inherits from E. Since p is a covering map, we can find a neighbourhood U of b which is evenly covered by p, i.e.,

$$p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha, \text{where the } V_\alpha \text{ are open.}$$

Since the union of the V_α is disjoint,

$$V_\alpha \cap p^{-1}(b) = \{x_\alpha\},$$

where x_α is the unique point in V_α such that $p(x_\alpha) = b$. We have just represented $\{x_\alpha\}$ as an intersection of an open set V_α with the subspace $p^{-1}(b) \subseteq E$, which shows that $\{x_\alpha\}$ is open in the subspace topology of $p^{-1}(b)$. Since

$$p^{-1}(b) = \bigsqcup_{\alpha \in A} \{x_\alpha\},$$

this shows $p^{-1}(b)$ is discrete.

§54. The Fundamental Group of the Circle

1. What goes wrong with the path-lifting lemma 54.1 for the local homeomorphism of Example 2 of §53?

   The very first step: you cannot find an open set U which contains $b_0 = (1, 0)$ and is evenly covered by p; p is not a covering map.

2. In defining the map $\tilde{F}$ in the proof of Lemma 54.2, why were we so careful about the order in which we considered the small rectangles?

   To maintain the continuity of the lift. If you performed the lifting for the squares in random order, there is no guarantee that the lifts would "connect" or "match up" along the boundaries of the little squares.

3. Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

   We need to show $p \circ (\tilde{\alpha} * \tilde{\beta}) = \alpha * \beta$. By definition, $p \circ (\tilde{\alpha} * \tilde{\beta}) : I \to E$ by

$$\begin{aligned} p \circ (\tilde{\alpha} * \tilde{\beta})(t) &= \begin{cases} p \circ \tilde{\alpha}(2t), & 0 \leq t \leq \frac{1}{2}, \\ p \circ \tilde{\beta}(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases} && \text{def of } * \\ &= \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases} && \text{def of } \tilde{\alpha}, \tilde{\beta} \\ &= \alpha * \beta(t) && \text{def of } *. \end{aligned}$$

4. Consider the covering map $p : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2_0$ of Example 6 of §53:[2]

$$(x, t) \mapsto ((\cos 2\pi x, \sin 2\pi x), t) \mapsto t(\cos 2\pi x, \sin 2\pi x).$$

Find liftings of the paths

$$f(t) = (2 - t, 0),$$
$$g(t) = ((1 + t)\cos 2\pi t, (1 + t)\sin 2\pi t),$$
$$h(t) = f * g.$$

Sketch the paths and their liftings.

The point here is that $f * g$ is a closed loop, but no lifting of it is.

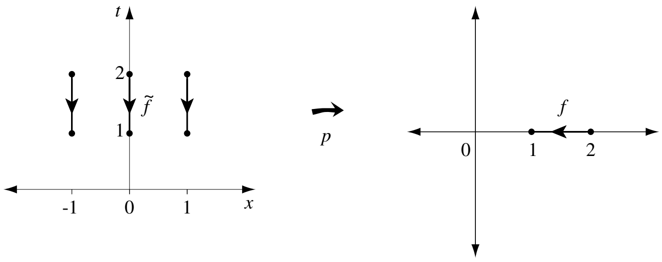

FIGURE 1. Three liftings of f. $\tilde{f}$ is the one in the centre.

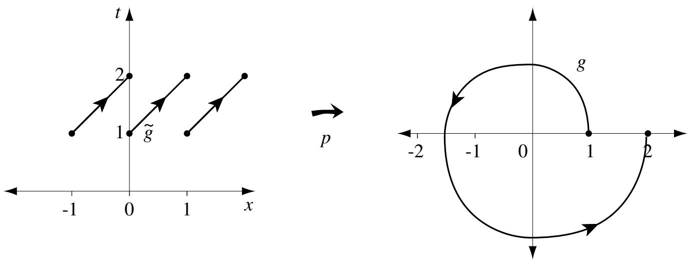

FIGURE 2. Three liftings of g. $\tilde{g}$ is the one in the centre.

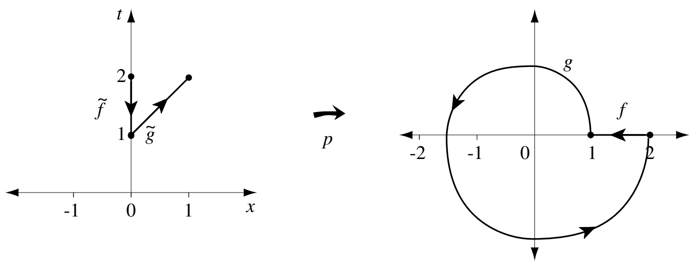

FIGURE 3. $\tilde{f} * \tilde{g}$ is a lifting of $f * g$.

[2] I use the shorthand $\mathbb{R}^2_0 = \mathbb{R}^2 \backslash \{(0, 0)\}$.

5. Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of §53. Consider the path in $S^1 \times S^1$ given by

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t) = (e^{it}, e^{2it}),$$

if we consider $\mathbb{T}^2 = S^1 \times S^1$ as a subspace of $\mathbb{C} \times \mathbb{C}$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

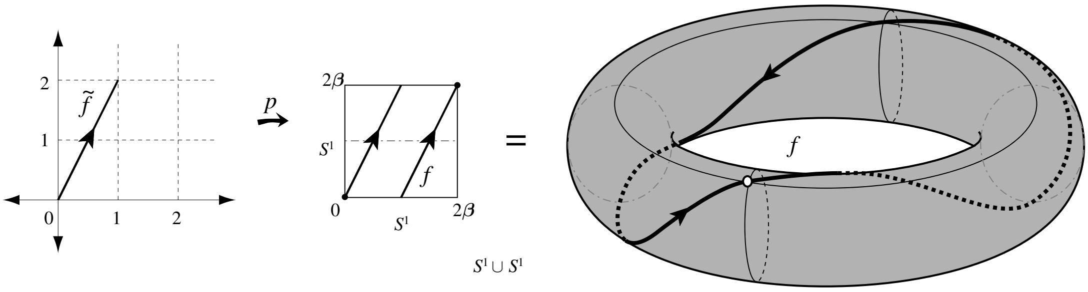

FIGURE 4. $\mathbb{T}^2 = S^1 \times S^1$ is represented two different ways on the right. The path (loop) f begins at the white dot and traverses $\mathbb{T}^2$. For visualization, the first S^1 axis is sketched on top of the doughnut, the second S^1 axis is sketched in front, with an extra copy in back to indicate where f goes "through the top" of the square representation of $\mathbb{T}^2$. Note that as the path traverses the horizontal S^1 once, it traverses the vertical S^1 twice.

6. Consider the maps $g, h : S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = z^{-n}$. Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself.

The group is cyclic, so it suffices to determine the image of its generator, $\gamma(t) = e^{2\pi i t}, t \in I$ under g. Since

$$g \circ \gamma(t) = g\left(e^{2\pi i t}\right) = \left(e^{2\pi i t}\right)^n = e^{2\pi i n t}$$

is a loop which goes n times around the circle, in the direction of γ, we have

$$g_*([\gamma]) = [g \circ \gamma] = \underbrace{[\gamma] * \cdots * [\gamma]}_{n \text{ times}} = [\gamma]^{*n} \qquad \text{or} \qquad g : \gamma(t) \mapsto \gamma(nt).$$

Similarly for h, we have that the loop

$$h \circ \gamma(t) = h\left(e^{2\pi i t}\right) = \left(e^{2\pi i t}\right)^{-n} = e^{-2\pi i n t}$$

goes n times around the circle, in the direction opposite to γ. This gives

$$h_*([\gamma]) = [h \circ \gamma] = \underbrace{[\overline{\gamma}] * \cdots * [\overline{\gamma}]}_{n \text{ times}} = [\overline{\gamma}]^{*n} = [\gamma]^{*(-n)} \qquad \text{or} \qquad h : \gamma(t) \mapsto \gamma(-nt).$$

7. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Let $p : \mathbb{R} \to S^1$ by the covering map given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Then by Thm 53.3, we may define

$$P = p \times p : \mathbb{R}^2 \to \mathbb{T}^2,$$

the standard cover of the torus, as in Example 4 of §53. Take $e_0 = (0, 0)$ and $b_0 = P(e_0)$. Then

$$P^{-1}(b_0) = \{(x, y) \in \mathbb{R}^2 \,\vdots\, x, y \in \mathbb{Z}\} = \mathbb{Z}^2.$$

Since $\mathbb{R}^2$ is simply connected, the lifting correspondence ϕ gives a bijection (by Thm 54.4), so we just need to show ϕ is a homomorphism.

Given $[f], [g] \in \pi_1(\mathbb{T}^2, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in $\mathbb{R}^2$ beginning at $e_0 = (0, 0)$. Let $(m, n) = \tilde{f}(1)$ and $(j, k) = \tilde{g}(1)$. Since these points are in the preimage of b_0, we know $m, n, j, k \in \mathbb{Z}$. Then

$$\phi([f]) + \phi([g]) = (m, n) + (j, k) = (m + j, n + k) \in \mathbb{Z}^2.$$

Now take $\tilde{\tilde{g}} = (m, n) + \tilde{g}$ to be the translate of $\tilde{g}$ which begins at $(m, n) \in \mathbb{R}^2$. Then

$$\begin{aligned} P \circ \tilde{\tilde{g}}(t) &= P((m, n) + \tilde{g}(t)) \\ &= (p(m + \tilde{g}_1(t)), p(n + \tilde{g}_2(t))) \\ &= (p(\tilde{g}_1(t)), p(\tilde{g}_2(t))) \\ &= P \circ \tilde{g}(t) \\ &= g(t), \end{aligned}$$

so $\tilde{\tilde{g}}$ is a lifting of g. The third equality here follows because $p(m + x) = p(x)$ by the formula for p, and the last line follows because $\tilde{g}$ is a lifting of g. Then the product $\tilde{f} * \tilde{\tilde{g}}$ is a well-defined path, and it is the lifting of $f * g$ that begins at $(0, 0)$. It is clear that it begins at $(0, 0)$, since f does. We check that it is a lifting:

$$\begin{aligned} P \circ \left(\tilde{f} * \tilde{\tilde{g}}\right)(t) &= \begin{cases} P(\tilde{f}(2t)), & 0 \le t \le \frac{1}{2}, \\ P(\tilde{\tilde{g}}(2t - 1)), & \frac{1}{2} \le t \le 1, \end{cases} && \text{def of } * \\ &= \begin{cases} f(2t), & 0 \le t \le \frac{1}{2}, \\ g(2t - 1)), & \frac{1}{2} \le t \le 1, \end{cases} && \text{by above} \\ &= f * g(t). \end{aligned}$$

The end point of $\tilde{\tilde{g}}$ is

$$\tilde{\tilde{g}}(1) = (m, n) + \tilde{g}(1) = (m + j, n + k).$$

Since $[f] * [g]$ begins at b_0, $\phi([f] * [g])$ is well-defined, and we have

$$\phi([f] * [g]) = \tilde{f} * \tilde{\tilde{g}}(1) = (m + j, n + k).$$

Thus $\phi([f] * [g]) = \phi([f]) + \phi([g])$, and we are done.

8. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then p is a homeomorphism.

By Thm 54.6(a), $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is 1-1. Since $\pi_1(B, b_0)$ is trivial by hypothesis, it must also be the case that $\pi_1(E, e_0)$ is also trivial, and p_* is actually an isomorphism. Then every loop f at b_0 is in H, and hence lifts to a loop $\tilde{f}$ at e_0, which shows $p^{-1}(b_0) = e_0$ by the lifting correspondence — if you have a surjection (by Thm 54.4) where the domain has only one element, then the range must also consists of only one element. Thus by Ex 53.3, E is a 1-fold covering of B, i.e., E and B are homeomorphic.

§55. Retractions and Fixed Points

Note: I use the following notation for the image of a map $f : X \to Y$.

$$\text{Im}(f) = \{y \in Y : y = f(x) \text{ for some } x \in X\}$$

1. Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

   If A is a retract, then $A \subseteq B^2$ and there is a retraction i.e., a continuous map $r : B^2 \to A$ with $r(a) = a$, $\forall a \in A$. Let $j : A \to B^2$ by inclusion. Then $j \circ f \circ r : B^2 \to B^2$ is a continuous map, so it has a fixed point by Brouwer's Theorem (55.6). Denote it by c. Since $Im(f \circ r) \subseteq A$, it must be that $c \in A$. Then

   $$\begin{aligned} c = j \circ f \circ r(c) &= j \circ f(c) && r\big|_A \text{ is the identity} \\ &= f(c) && j \text{ is inclusion.} \end{aligned}$$

2. Suppose $h : S^1 \to S^1$ is nulhomotopic.

   (a) Show that h has a fixed point.

   h extends to $k : B^2 \to S^1$ by Lemma 55.3 with $X = S^1$. But $S^1 \subseteq B^2$, so we can actually consider k as a mapping $k : B^2 \to B^2$. Then k must have a fixed point by Brouwer's Theorem (55.6), call it b. Since $\text{Im}(k) \subseteq S^1$, $k(b) = b$ implies that $b \in S^1$. Thus $h(b)$ is defined. In fact,

   $$\begin{aligned} h(b) &= k(b) && \text{as an extension, } k = h \text{ on } S^1 \\ &= b && b \text{ is a fixed point of } k. \end{aligned}$$

   (b) Show that h maps some point x to its antipode $-x$.

   Define $f : S^1 \to S^1$ by $f(x) = -x$. Then $f \circ h : S^1 \to S^1$ is nulhomotopic, so it must have a fixed point by (a), call it a. Then

   $$\begin{aligned} a &= f \circ h(a) = f(h(a)) = -h(a) \\ \implies h(a) &= -a. \end{aligned}$$

3. Show that if A is a nonsingular 3×3 matrix having nonnegative entries, then A has a positive real eigenvalue.

   Following Cor. 55.7, let $B = S^2 \cap O_1 \subseteq \mathbb{R}^3$. All the components of $x \in B$ are nonnegative, and $x \neq 0$. Since A is nonsingular, $Ax \neq 0$, and the map

   $$S : B \to B \quad \text{by} \quad S(x) = Ax/\|Ax\|$$

is well-defined. Since B is homeomorphic to the ball, S must have a fixed point x_0 by Brouwer's Thm (55.6). Then

$$x_0 = Ax_0/\|Ax_0\| \quad \Longrightarrow \quad Ax_0 = \|Ax_0\|x_0,$$

i.e., A has an eigenvector x_0 with corresponding eigenvalue $\|Ax_0\| > 0$.

4. Assume: for each n, there is no retraction $r : B^{n+1} \to S^n$.

   (a) The identity map $i : S^n \to S^n$ is not nulhomotopic.

   Suppose $i = id_{S^n}$ were nulhomotopic. We follow the proof of Lemma 55.3, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Let $H : S^n \times I \to S^n$ be a homotopy between i and a constant map. Define $\pi : S^n \times I \to B^{n+1}$ by

   $$\pi(x, t) = (1 - t)x.$$

   Then π is a quotient map and H induces a continuous map $k : B^{n+1} \to S^n$ (via π) that is an extension

   (b) The inclusion map $j : S^n \to \mathbb{R}^{n+1}$ is not nulhomotopic.

   (c) Every nonvanishing vector field on B^{n+1} points directly outward at some point, and directly inward at some point.

   (d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

   (e) Every $(n + 1) \times (n + 1)$ matrix with positive real entries has a positive real eigenvalue.

   (f) If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point, and h maps some point x to its antipode $-x$.

§58. Deformation Retracts and Homotopy Type

Note: I am occasionally sloppy and saying "x_0" when I mean "the constant map at x_0". This is a standard abuse of language.

1. Show that if A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

   So we have some $r : X \to A$ which is the identity on A, and $s : A \to B$ which is the identity on B. Then $s \circ r : X \to B$ and for $b \in B$,

   $$s \circ r(b) = s(r(b)) = s(b) = b,$$

   because $b \in B \subseteq A \subseteq X$. We also have $r \simeq_H id_X$ by some homotopy H which keeps every point of A fixed. In particular, H keeps every point of $B \subseteq A$ fixed. Also, we have $s \simeq_K id_A$ by some homotopy K which keeps every point of B fixed. Define $F : X \times I \to X$ by

   $$F(x,t) = \begin{cases} H(x,2t), & 0 \leq t \leq \frac{1}{2}, H(x,0) = id_X, H(x,1) = r(x), \\ K(r(x), 2t-1), & \frac{1}{2} \leq t \leq 1, K(x,0) = id_A, K(x,1) = s(x), \end{cases}$$

   as in §51 Exercise 1. Then $s \circ r \simeq_F id_X$.

2. For each of the following spaces, the fundamental group is either (1) trivial, (2) infinite cyclic, or (3) isomorphic to the fundamental group of the figure eight.

   (a) The solid torus $B^2 \times S^1$.
   For $(x,y) \in B^2 \times S^1$, the homotopy

   $$H((x,y),t) = ((1-t)x, y)$$

   shows that S^1 is a deformation retract of the solid torus, so $\pi_1(B^2 \times S^1) = \pi_1(S^1) = \mathbb{Z}$.

   (b) The torus with a point removed.
   $I^1 \backslash \{(\frac{1}{2}, \frac{1}{2})\}$ has its boundary as a deformation retract, by the straight-line homotopy (from $(\frac{1}{2}, \frac{1}{2})$). Under the quotient map which makes I^2 into the torus, $\partial \left(I^1 \backslash \{(\frac{1}{2}, \frac{1}{2})\}\right)$ becomes a figure-eight.

   (c) The cylinder $C = S^1 \times I$.
   $\pi_1(C) = \mathbb{Z}$, because the cylinder has S^1 as a deformation retract by the homotopy

   $$H((x,y),t) = (x, (1-t)y).$$

   (d) The infinite cylinder $IC = S^1 \times \mathbb{R}$.
   $\pi_1(IC) = \mathbb{Z}$ by the same reason (and same homotopy) as (c).

(e) $\mathbb{R}^3$ with the nonnegative axes deleted.
Consider that any loop in this space is homotopic to some combination of α, β, γ (and their inverses), as depicted in Figure 5. Note that $\alpha * \beta = \gamma$, so that the

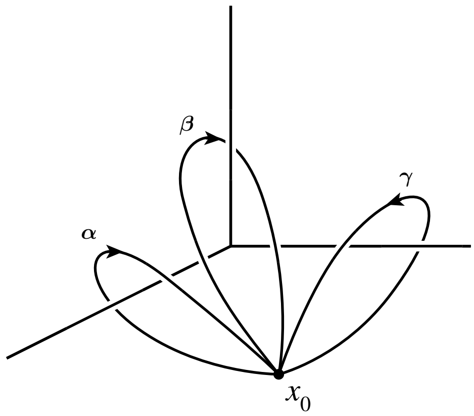

FIGURE 5. Loops around the axes in $\mathbb{R}^3$.

fundamental group can be generated without γ. Also, note that $\beta * \alpha \simeq \overline{\gamma} \neq \gamma$, so fundamental group is not abelian.

(f) $\{x : \|x\| > 1\}$.
This space has $2S^1 = \{x \in \mathbb{R}^2 : \|x\| = 2\}$ as a deformation retract by the straight-line homotopy $H(x,t) = (1-t)x + 2tx/\|x\|$. $2S^1$ is obviously homeomorphic to S^1 by $x \mapsto x/2$, so $\pi_1((\text{f})) = \mathbb{Z}$.

(g) $\{x : \|x\| \geq 1\}$.
This space has S^1 as a deformation retract by the straight-line homotopy $H(x,t) = (1-t)x + tx/\|x\|$, so $\pi_1((\text{g})) = \mathbb{Z}$.

(h) $\{x : \|x\| < 1\}$.
By $H(x,t) = (1-t)x$, $\pi_1((\text{h}))$ is trivial.

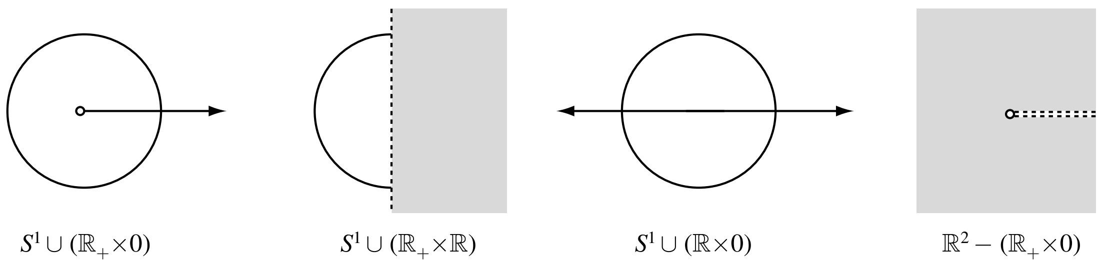

FIGURE 6. The spaces in exercises (i)-(l).

(i) $S^1 \cup (\mathbb{R}_+ \times 0)$.
$\pi_1((\text{i}))$ is trivial by $H(x,t) = (1-t)x + tx/\|x\|$.

(j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$.
$\pi_1((\text{j}))$ is trivial by $H(x,t) = (1-t)x + tx/\|x\|$.

(k) $S^1 \cup (\mathbb{R} \times 0)$.
By the homotopy

$$H(x,t) = \begin{cases} x & \|x\| \leq 1, \\ (1-t)x + tx/\|x\| & \|x\| > 1. \end{cases},$$

we obtain the theta space of Example 3. This has the fundamental group of the figure eight, as both are retracts of the doubly punctured plane (apply Thm. 58.3 to Example 2).

(l) $\mathbb{R}^2 \backslash (\mathbb{R}_+ \times 0)$.
By $H(x,t) = (1-t)x + t(-1,0)$, $\pi_1((\text{l}))$ is trivial.

3. Show that given a collection $\mathcal{C}$ of spaces, the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$.

We show the three properties.

(i) Let $f = g = id_X : X \to X$. Then $f \circ g, g \circ f$ are both homotopic to the identity on X (since they *are* the identity on X). So $X \approxeq X$.

(ii) Let $X \approxeq Y$. Then $\exists f : X \to Y, g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. But this is just the same as $Y \approxeq X$.

(iii) Let $X \approxeq Y$ and $Y \approxeq Z$. Then we have homotopy equivalences $f : X \to Y$ and $h : Y \to Z$, with corresponding inverses $g : Y \to X$ and $k : Z \to Y$. We need to show that $h \circ f : X \to Z$ and $g \circ k : Z \to X$ are homotopy inverses of each other. By associativity,

$$(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f.$$

Then h, k are homotopy inverses, so $k \circ h \simeq id_Y$ and

$$\begin{aligned} g \circ (k \circ h) &\simeq g \circ id_Y && \text{§51 Exercise 1} \\ g \circ (k \circ h) \circ f &\simeq g \circ id_Y \circ f(x) && \text{§51 Exercise 1 again} \\ &\simeq g \circ f(x) && \text{by trivial homotopy} \\ &\simeq id_X(x) && \text{Lemma 51.1 (transitivity).} \end{aligned}$$

The argument is identical for $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k$.

4. Let X be the figure eight and let Y be the theta space. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other.

Define the maps as indicated in Figure 7. f maps all the points inside the dotted line to the vertical bar of the theta. Note: it is not 1-1, so it cannot have an inverse. g

maps all the points of the vertical bar to the "centre point" of the figure-eight where the two loops connect. It is similarly noninjective and hence noninvertible. These

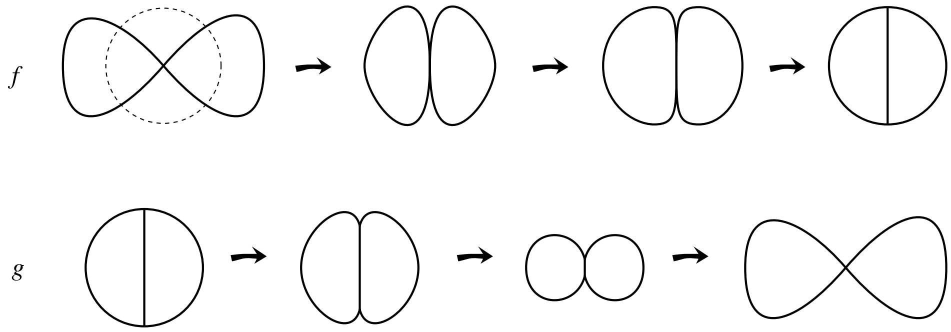

FIGURE 7. The spaces in exercises (i)-(l).

maps are not inverses, but they are homotopy inverses. There is a homotopy between id_X and $g \circ f$ which maps the figure eight to itself, but continuously scrunches all the points inside the dotted line down into the centre point, as t goes from 0 to 1. Similarly, there is a homotopy from id_Y to $f \circ g$ which acts by collapsing the vertical bar to the centre point of the vertical bar, and continuously slurping in part of the boundary of the circle to replace the bar.

5. Recall that a space X is *contractible* iff the identity map id_X is nulhomotopic. Show that X is contractible iff X has the homotopy type of a one-point space.

   This is just an application of Exercise 51.3(c) to $[X, X]$.

6. Show that a retract of a contractible space is contractible.

   If X is contractible, then we have $H : X \times I \to X$ with
$$H(x, 0) = x, \quad H(x, 1) = x_0, \qquad \forall x \in X.$$
   Suppose $r : X \to A$ is a retraction, so that $r\big|_A = id_A$. Since it doesn't matter what point x_0 we chose above (X is path connected by Exercise 51.3(b)), let $x_0 \in A$. Then $H\big|_{A\times I}$ defines a homotopy from id_A to x_0.

§59. The Fundamental Group of S^n

1. Let X be the union of two copies of S^2 having a single point in common. What is the fundamental group of X?

   Consider X as sitting in $\mathbb{R}^3$ along the x-axis, for purposes of description. The projection of X to the x-axis would then be $[-2, 2]$, with the single point in common projecting to 0. Let

   $$U = X \cap (-1, \infty) \times \mathbb{R}^2 \quad \text{and} \quad V = X \cap (-\infty, 1) \times \mathbb{R}^2,$$

   so that U and V are clearly open. U has a copy of S^2 as a deformation retract by collapsing the left hemisphere along the meridians leading to the origin, so $\pi_1(U) = \pi_1(S^n)$ is trivial. Similarly for V. $U \cap V$ is clearly nonempty and contractible, hence path-connected, so Cor. 59.2 applies and $\pi_1(X)$ is trivial.

2. Criticize the following "proof" that S^2 is simply connected: Let f be a loop in S^2 based at x_0. Choose a point $p \in S^2$ not lying in the image of f. Since $S^2 \backslash p$ is homeomorphic with $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected, the loop f is path homotopic to the constant loop.

   You may not be able to pick a point p in the image of f. It is possible to have a continuous surjection from I to S^2 (of course, the inverse will not be continuous). Consider a composition with the the Peano map (see Thm. 44.1 on p. 272).

3. (I use the shorthand $\mathbb{R}^n_0 = \mathbb{R}^n \backslash \{0\}$.)

   (a) Show that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

   $\mathbb{R}_0 = (-\infty, 0) \cup (0, \infty)$ is not connected, but $\mathbb{R}^n_0$ is. In fact, it is path connected. For $x, y \in \mathbb{R}^n_0$, use the straight line path $\overline{xy}$, unless $(1-t)x + ty = 0$ for some $t \in [0, 1]$. In this case, pick any $z \in \mathbb{R}^n_0$ that does not lie on the line through x, y and take the straight line path $\overline{xz}$ followed by the straight line path $\overline{zy}$.
   Now suppose we had a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$. Then the restriction of f to $\mathbb{R}_0$ would give a homeomorphism between a connected and a disconnected set. ↯

   (b) Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$.

   The solution is similar to the previous problem, but we use simply connected instead of connected. For every n, $\mathbb{R}^n_0$ has S^{n-1} as a deformation retract by

   $$H(x, t) = (1-t)x + tx/\|x\|.$$

   For $n = 2$ this gives

   $$\pi_1(\mathbb{R}^2_0) = \pi_1(S^1) = \mathbb{Z}.$$

For $n \geq 3$, this gives

$$\pi_1(\mathbb{R}^n_0) = \pi_1(S^{n-1}) = 0.$$

Since they have different fundamental groups, they cannot be homeomorphic. This is the essential point of this course, as presented formally (and more strongly) in Thm. 58.7.

4. Assume the hypotheses of Theorem 59.1.

   (a) What can you say about the fundamental group of X if j_* is the trivial homomorphism? If both i_* and j_* are trivial?

   If both are trivial, then the image of each is just the identity element. By Theorem 59.1, $\pi_1(X)$ will be the group generated by the identity; but that's just the trivial group.
   If just j_* is trivial, there's not much you can say. It is not hard to come up with cases where U does not have a trivial fundamental group, and neither does X. For example, let

   $$U = \{(x, y, z) \in S^2 : -1 < x < 0\},$$

   so that U is a punctured hemisphere, and let V be S^2 with two punctures, each of which has positive x-coordinate.

   (b) Give an example where i_* and j_* are trivial, but neither U nor V have trivial fundamental groups.

   Let U be S^2 with two punctures, each of which has negative x-coordinate. Let V be S^2 with two punctures, each of which has positive x-coordinate. Then each of U, V has a copy of S^1 as a deformation retract, and thus has fundamental group $\mathbb{Z}$. However, $X = U \cup V$ is S^2, which has trivial fundamental group. Thus, each of i_*, j_* must send every loop class to the identity class (since there's nothing else in $\pi_1(S^2)$ to send it to) and is hence a trivial homomorphism.

§60. The Fundamental Group of Some Surfaces

1. Compute the fundamental group of the solid torus $S^1 \times B^2$ and the product space $S^1 \times S^2$.

   Applying Thm. 60.1,

$$\pi_1(S^1 \times B^2) \cong \pi_1(S^1) \times \pi_1(B^2) \cong \mathbb{Z} \times \{e\} \cong \mathbb{Z}, \text{ and}$$
$$\pi_1(S^1 \times S^2) \cong \pi_1(S^1) \times \pi_1(S^2) \cong \mathbb{Z} \times \{e\} \cong \mathbb{Z}.$$

2. Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to the projective plane P^2.

   An element of P^2 looks like $\{x, -x\}$ where $x \in S^2$. Consider the copy of B^2 which is the projection of S^2 into the xy-plane, and let $\sim$ be the equivalence relation which identifies antipodal points of B^2. Define a map $f : P^2 \to B^2/\sim$ by

$$f(\{x,-x\}) = \begin{cases} p(x) & z(x) \geq 0 \\ p(-x) & z(x) \leq 0. \end{cases},$$

where $p(x)$ is the orthogonal projection of x onto the xy-plane, and $z(x)$ is the third coordinate of x. In other words, if $x \in S^2$ lies above the xy-plane, then map $\{x, -x\}$ onto the projection of x, and if $x \in S^2$ lies below the xy-plane, then map $\{x, -x\}$ onto the projection of $-x$.

Projection is continuous, so we can use the Pasting Lemma if we verify that f is continuous for x such that $z(x) = 0$. But for such an x, $p(x) = p(-x)$ by the equivalence relation, since $p(x)$ and $p(-x)$ are antipodal points of S^2.

Now note that f is bijective, by constructing the inverse. For $b \in B^2/\sim$, let

$$x = S^2_+ \cap \{b + (0,0,t) : t \in \mathbb{R}\}$$

be the point in the upper hemisphere of S^2 which lies in the vertical line through b, and then define

$$g(b) = \{x, -x\}.$$

Now by construction,

$$f \circ g(b) = f(\{x,-x\}) = p(x) = b, \quad \text{and}$$

$$g \circ f(\{x,-x\}) = \begin{cases} g(p(x)) & z(x) \geq 0 \\ g(p(-x)) & z(x) \leq 0 \end{cases} = \begin{cases} \{x,-x\} & z(x) \geq 0 \\ \{-x,x\} & z(x) \leq 0 \end{cases} = \{-x, x\}.$$

Note that $B^2/\sim$ is compact, by Compactness Mantra (1), and that P^2 is Hausdorff by Thm. 60.3. Now apply the following lemma to f^{-1}, and conclude that f^{-1} (and hence also f) is a homeomorphism.

Lemma. If $f : X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.

Proof. We need to show f is open. Let $U \subseteq X$ be open, so that U^C is closed. f is a closed map, by Exercise 2 from the Fundamental Mantras of Compactness, so $f(U^C)$ is closed. Now using the Basic Survival Tools #3(d),

$$\begin{aligned} f(U^C) &= f(X \backslash U) && \text{def complement} \\ &= f(X) \backslash f(U) && \text{BST 3(d)} \\ &= Y \backslash f(U) && f \text{ is onto} \\ &= f(U)^C, \end{aligned}$$

so $f(U)$ is open. □

3. Let $p : E \to X$ be the map constructed in the proof of Lemma 60.5. Let E' be the subspace of E that is the union of the x-axis and the y-axis. Show that $p\big|_{E'}$ is not a covering map.

   Consider a tiny open disc centered at x_0. Its intersection with X is a small open "X" shape. Its preimage looks like a similar open "X" centered at the origin, as well as a small open horizontal interval around every point $(n, 0)$ and a small vertical interval around every point $(0, n)$, where $n \in \mathbb{Z}$.

   $p\big|_{E'}$ is not a covering map because while these open intervals are open and disjoint, they are not homeomorphic to the "X" in the base space. To see this, note that any map from an "X" onto an interval is necessarily not injective.

4. The space P^1 and the covering map $p : S^1 \to P^1$ are familiar ones. What are they?

   For $z \in S^1$, define

$$p(z) = z^2.$$

   Check that this maps antipodal points to the same point:

$$p(-z) = (-z)^2 = z^2 = p(z).$$

   We already know this is a covering map, by previous problems (and homework). So $P^1 = S^1$.

5. Consider the covering map indicated in Figure 8. Here, p wraps A_1 around A twice and wraps B_1 around B twice; p maps A_0 homeomorphically onto A and B, respectively. Use this covering space to show that the fundamental group of the figure eight is not abelian.

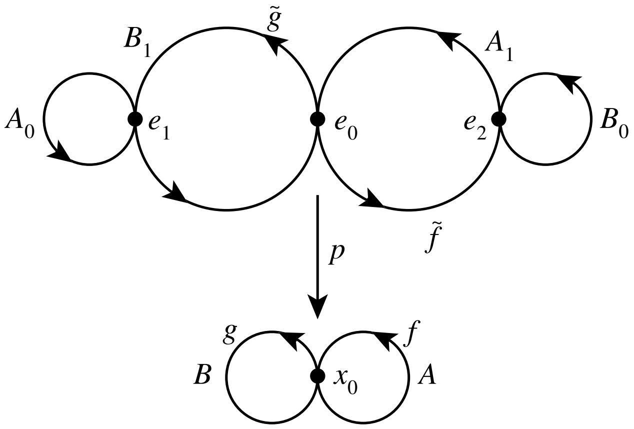

FIGURE 8. An alternative cover for the figure-eight.

Let f loop once around A counterclockwise and let g loop once around B counterclockwise. By Thm. 54.3,

$$f * g \simeq_p g * f \quad \Longleftrightarrow \quad (f * g)^{\sim}, (g * f)^{\sim} \quad \text{have the same endpoint.}$$

The lifting of f is $\tilde{f}$ running counterclockwise from e_0 to e_2, so the lifting of $f * g$ looks like $\tilde{f}$ followed by a counterclockwise loop around B_0 ending at e_2.

The lifting of g is $\tilde{g}$ running counterclockwise from e_0 to e_1, so the lifting of $g * f$ looks like $\tilde{g}$ followed by a counterclockwise loop around A_0 ending at e_1.

Since these liftings have different endpoints, Thm. 54.3 indicates that $f * g$ and $g * f$ are not path homotopic. Hence

$$[f] * [g] = [f * g] \neq [g * f] = [g] * [f],$$

and the fundamental group is not abelian.