Introduction to mathematical Statistics **Quiz 5 Solution**

1. (34 points) Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, be a random sample from the normal population. Please derive the best estimator for μ assuming that σ^2 is known.

Solution:

First, using the moment generating function technique, we can show that the sample mean $\bar{X}$ (both an MLE and an MOME as shown in Quiz 4) is an unbiased estimator of μ, furthermore, its distribution is: $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Since the normal population satisfies all regularity conditions, we now derive the Cramer-Rao lower bound for the variance of an unbiased estimator of μ as follows:

$$f_X(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}}$$

$$\ln f = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{(X_i-\mu)^2}{2\sigma^2}$$

$$\frac{d \ln f}{d\mu} = \frac{(X_i-\mu)}{\sigma^2}$$

$$\frac{d^2 \ln f}{d\mu^2} = -\frac{1}{\sigma^2}$$

Hence, the Cramer-Rao lower bound for the variance of an unbiased estimator of μ is $\frac{\sigma^2}{n}$. Since the variance of $\bar{X}$ equals to the C-R lower bound, and the normal population satisfies all regularity conditions necessary for the C-R lower bound theorem to hold, we declare that the variance of $\bar{X}$ is the lowest an unbiased estimator for μ can attain, and thus $\bar{X}$ is both an efficient and a best estimator for μ

2. (33 points) Suppose a random sample of size n is drawn from a population with pdf $f(y;\theta) = \exp(\theta - y), y \geq \theta > 0$; (and 0 elsewhere). Please find the MLE for θ.

Solution:

$L(\theta) = \prod_{i=1}^{n} \exp(\theta - y_i) = \exp(n\theta - n\bar{y}), for\ y_{(n)} \geq \cdots \geq y_{(1)} \geq \theta > 0$; and 0 elsewhere

So the MLE is $\hat{\theta} = Y_{(1)} = \min(Y_1, \ldots, Y_n)$

3. (33 points) Suppose that Y_1=0.42, Y_2=0.10, Y_3=0.64, Y_4=0.23 is a random sample of size four from the p.d.f:

$$f_Y(y;\theta) = \theta y^{\theta-1} \qquad 0 \leq y \leq 1$$

Find the MOM estimate for θ.

Solution: First, we calculate the theoretical first moment:

$$EY = \int_0^1 y\theta y^{\theta-1} dy = \frac{\theta}{\theta+1} y^{\theta} \Big|_0^1 = \frac{\theta}{\theta+1}$$

Second, we calculate the first sample moment:

$$\frac{\sum Y_i}{n} = \bar{y} = \frac{0.42+0.10+0.64+0.23}{4} = 0.35$$

Third, set them to be equal:

$$\frac{\theta}{\theta+1} = \overline{y} = 0.35$$

The resulting MOME is $\hat{\theta} = 0.54$

Homework #3. Due Friday, March 18, before class. (Ch. 5.)

5.2.1, 5.2.2, 5.2.3, 5.2.5, 5.2.8, 5.2.11, 5.3.1, 5.3.2, 5.3.3, 5.3.7, 5.3.10, 5.3.16, 5.4.7, 5.4.9, 5.4.10, 5.4.11, 5.5.11, 5.5.14, 5.5.18 (19 problems in total)