Introduction to mathematical Statistics **Quiz 4 Solution**

1. Suppose a random sample of size n is drawn from a normal population with mean μ and variance σ^2, where σ^2 is known. Please.

(a). Find the maximum likelihood estimators of μ

(b). Find the method of moment estimators of μ

Solutions 1. (a)

[i] $f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right], x_i \in R, i = 1, \ldots, n$

[ii] likelihood function

$$L = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \{(2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]\} = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right].$$

[iii] log likelihood function

$$l = \ln L = \left(-\frac{n}{2}\right)\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

[iv]

$$\frac{dl}{d\mu} = \frac{2\sum_{i=1}^{n}(x_i-\mu)}{2\sigma^2} = 0$$

$$\Rightarrow \hat{\mu} = \bar{X}$$

1. (b) $E(X) = \mu = \bar{X}$

$$\Rightarrow \hat{\mu} = \bar{X}$$

2. Let X_i, i = 1, …, n, denote the outcome of a series of n independent trials, where $X_i = 1$ with probability p, and Xi = 0 with probability (1- p). Let $W = \sum_{i=1}^{n} X_i$ and $\hat{p} = \frac{W}{n}$.
(a). Please derive the method of moment estimator of p.
(b). Please derive the maximum likelihood estimator of p.

Solutions 2. (a)

$$X_i \overset{i.i.d.}{\sim} \text{Bernoulli}(p),$$
$$E(X_i) = 1 * p + 0 * (1-p) = p,$$
$$Var(X_i) = E(X_i^2) - [E(X_i^2)] = 1^2 * p + 0^2 * (1-p) - [p]^2 = p - p^2 = p(1-p)$$

Therefore the first population mean is $E(X_i)=p$
And the first sample mean is: $\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{W}{n}$
Set them to be equal and we found the moment estimator of p to be: $\hat{p} = \frac{W}{n}$.

2. (b)

[i] $f(x_i) = p^{x_i}(1-p)^{1-x_i}, i = 1, \dots, n$
[ii] $L = \prod f(x_i) = p^{\sum x_i}(1-p)^{n-\sum x_i}$
[iii] $l = \ln L = (\sum x_i) \ln p + (n - \sum x_i) \ln(1-p)$
[iv] $\frac{dl}{dp} = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} = 0$

$\Rightarrow \hat{p} = \frac{\sum X_i}{n} = \frac{W}{n}$ is the MLE of p