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A Introduction to mathematical Statistics <u>Quiz 3 Solution</u>

1. (a) Let X~Bernoulli(p), that is, X can take on two possible values, 1 and 0, with probabilities p and (1-p) respectively, please derive its moment generating function. (b) Let $X_i \sim \text{Bernoulli}(p)$, $i = 1, \cdots, n$. Furthermore, the X_i's are independent to each other. Please derive the moment generating function of $Y = \sum_{i=1}^{n} X_i$. Do you recognize the distribution of Y?

Solution:

$$\text{(a) } M_X(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} P(X = x) = e^{t*0}(1-p) + e^{t*1}p = pe^t + (1-p)$$

$$\text{(b) } M_Y(t) = E(e^{tY}) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = E\left(\prod_{i=1}^{n} e^{tX_i}\right) = \prod_{i=1}^{n} E(e^{tX_i}) = [pe^t + (1-p)]^n$$

$\therefore Y \sim B(n, p)$

2. The gunner on a small assault boat fires six missiles at an attacking plane. Each has a 20% chance of being on target. If two or more of the shells find their mark, the plane will crash. At the same time, the pilot of the plane fires 10 air-to-surface rockets, each of which has a 0.05 chance of destroying the boat. Would you rather be on the plane or the boat? (That is, please calculate and compare the probability that the plane will crash and the probability that the boat will be destroyed.)

Solution:

Let X_1 = the number of missile hits on the plane. Then X_1 is binomial with $n = 6$ and $p = 0.2$. The probability the plane will crash is $P(X_1 \geq 2) = 1 - P(X_1 = 0) - P(X_1 = 1) = 1 - (0.8)^6 - 6(0.2)(0.8)^5 = 1 - 0.262 - 0.393 = 0.345$.

Let X_2 = the number of rocket hits on the boat . Then X_2 is binomial with $n = 10$ and $p = 0.05$. The probability the boat will be disabled is $P(X_2 \geq 1) = 1 - P(X_2 = 0) = 1 - (0.95)^{10} = 1 - 0.599 = 0.401$.

3. Suppose that the life of a certain light bulb is exponentially distributed with mean 100 hours. If 10 such light bulbs are installed simultaneously, what is the distribution of the life of the light bulb that fails last? Please show the entire derivation. *Note, the p.d.f. of the exponential random variable X with parameter λ is: $f(x) = \lambda e^{-\lambda x},\ x \succ 0$

Solution:

Let $X_1 \ldots X_{10}$ denote the life of the 10 bulbs respectively. Let Y denote the life of the bulb that fails last. Then $X_1 \ldots X_{10}$ are i.i.d. Exp(1/100) and Y = max { $X_1 \ldots X_{10}$ }.

$F_Y(y) = P(Y \leq y) = P(\max\{X_1 \ldots X_{10}\} \leq y)$

$= P(X_1 \leq y, \ldots, X_{10} \leq y)$

$= P(X_1 \leq y) \times \ldots \times P(X_{10} \leq y)$

$= (1 - e^{-\frac{y}{100}})^{10}$

And the PDF is

$f_Y(y) = \frac{1}{10} e^{-\frac{y}{100}} (1 - e^{-\frac{y}{100}})^9$

where y≥0.

4. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability 0.7, whereas when the defendant is, in fact, innocent, this probability drops to 0.2. If 70 percent of defendants are guilty, compute the probability that

(a) the jury would render a correct decision;
(b) an innocent man would be found innocent.

Solution:

Define event as follows

FG – the defendant is in fact guilty

VG – the defendant is voted guilty by one judge

FI – the defendant is in fact innocent

VI – the defendant is voted innocent by one judge

P{FG} = 0.7 P{VG|FG} = 0.7 P{VG|FI} = 0.2. So P{FI} =0.3.

Let X be the number of votes of guilty cast.

(a). If the defendant is in fact guilty, X follows the binomial distribution Bi(3, P{VG|FG}), i.e., Bi(3, 0.7). In this case

P{declared guilty | FG} = P(X=3)+P(X=2)= $\binom{3}{3}0.7^3 + \binom{3}{2}0.7^2 0.3$ =0.784

If the defendant is in fact innocent, X follows the binomial distribution Bi(3, P{VG|FI}), i.e., Bi(3, 0.2). In this case

P{declared guilty | FI} = P(X=3)+P(X=2)= $\binom{3}{3}0.2^3 + \binom{3}{2}0.2^2 0.8$ =0.104

So P{declared innocent | FI} = 1 – 0.104 = 0.896

P{Correct decision} = P{declared guilty | FG} P{FG} + P{declared innocent | FI} P{FI}

=0.784×0.7+0.896*0.3

=0.8176

(b). As shown above

P{declared innocent | FI} = 1 – 0.104 = 0.896

5. (extra credit for all). What is the probability that the larger of two random observations drawn from any continuous pdf will exceed the sixtieth percentile?

Solution:

$$P(Y_2' > y_{60}) = 1 - P(Y_2' < y_{60}) = 1 - P(Y_1 < y_{60}, Y_2 < y_{60})$$
$$= 1 - P(Y_1 < y_{60})P(Y_2 < y_{60}) = 1 - (0.60)(0.60) = 0.64$$

*****That's all, folks!*****